Graduate Quantum Mechanics - Problem Set 9

Problem 1)

A particle in 2 dimensions is described by a wave function $\Psi(x,y)$. We can make a variable substitution to circular (cylindrical) variables $(x,y) \to (r, \varphi)$ by defining an alternative wave function in terms of the new variables which describes the exact same state: $\Phi(r,\varphi) = \Psi(x,y) = \Psi(r \cos\varphi, r \sin\varphi)$. Using the definition of $\mathbf{L}_z$ in position (x,y) space, show that its representation in terms of these new variables is $\mathbf{L}_z = \frac{\hbar}{i}\frac{\partial}{\partial\varphi}$. Hint: Simply apply this form of L_z to $\Phi(r,\varphi)$ and show that you get the same answer as applying $\mathbf{L}_z$ in Cartesian coordinates to $\Psi(x,y)$.

Problem 2)

a) A particle of mass m is constrained to move in a circle of fixed radius R around the origin in the x-y plane in the absence of any potential energy. Using the generalized coordinates $q = \varphi$ and $p_\varphi = L_z$, write down the classical Hamiltonian for this system.
b) Now assume that the same system in quantum mechanics is governed by a Hamiltonian operator of the same functional form, i.e., by replacing L_z with the operator $\mathbf{L}_z$. Find the eigenfunctions $\Phi_i(\varphi)$ and the eigenvalues E_i of this Hamiltonian (using the results from Problem 1). Make sure your solutions are "single-valued" (meaning for the same physical point in space, the eigenfunctions have the same value).
c) What can you say about the difference in energy of any two "adjacent" eigenfunctions? What regular pattern would you observe in the frequency spectrum of photons emitted by this system when it undergoes a sequence of transitions from some high energy eigenstate to the next lower one and so on all the way to the ground state? (Draw a graph of the photon energy spectrum that one would obtain)
d) Which kinematic quantities (operator expectation values) are conserved (independent of time) for this system (even when it is **not** in an eigenstate of the Hamiltonian)? Which are **not** conserved? List two examples of each kind!

Hint: Show, as a first step, that the solutions to the stationary Schrödinger equation can be chosen to be simultaneous eigenfunctions of $\mathbf{L}_z$.
Note: This example is somewhat problematic in that the radial part of the wave function is nonsensical/inconsistent with the laws of quantum mechanics. However, you can simply ignore the radial behavior and write the solutions as functions of φ only. Keep in mind, though, that they must be single-valued for any physical space in point (i.e., have the same value for $\varphi + 2\pi$ as for φ).

Problem 3)

Show that the translational operator introduced in lecture:

$$T(\Delta x) = \exp\left(-i\frac{\Delta x P}{\hbar}\right)$$

applied to a state $|\psi\rangle$ leads, in the x-basis, to the same result as a Taylor expansion of the function $\psi(x-\Delta x)$ around the point $\Delta x = 0$.