Vector Space

consists of two parts: actual space of vectors, and the corresponding field of scalars (numbers)

- Scalars = real numbers $\mathbb{R}$.

  Spatial Vectors in 2D, 3D, ... nD space

  4D Minkowski space

  Complex numbers as pairs of real numbers: $a + ib; a, b \in \mathbb{R}$

- Scalars = complex numbers $\mathbb{C}$

  Ex. 1: column vectors ($c_i \in \mathbb{C}$)

$$\begin{pmatrix} c_1 \\ ... \\ c_n \end{pmatrix}$$

  Ex. 2: All functions of type $f(x) : x \in [0, L] \to \mathbb{C}$

ket: $|v>$ is an abstract way to write any vector. Don't confuse with *representation* of a given vector through a column of numbers (see below)

all vector spaces contain $|0>$; if a vector space contains $|v>$, it must also contain $|-v>$; $|v> + |-v> = |0>$

Linearly independent:

A set of vectors v_i is linearly independent if $\Sigma_{i=1}^n a_i |v_i> \neq 0$ unless $a_i = 0 \forall i \in [1, n]$

If this holds for a set of n vectors, but doesn't hold for *any* set of $n + 1$ vectors $\Rightarrow$ the space has n dimensions

basis set: set of vectors $|v_1> ... |v_n>$ in n-dimensional vector space that are linearly independent

Given arbitrary vector $|u>$, there *must* be numbers a_i, c such that $\Sigma_{i=1}^n [a_i |v_i> + c|u>] = 0$

$\Rightarrow |u> = -\Sigma_{i=1}^n \frac{a_i}{c} |v_i>$. We can use the numbers $-\frac{a_i}{c}$, written as a column, as a *representation* of the vector $|u>$ with respect to this particular basis.

Example: Complex vector space of 2x2 Matrices

4 dimensions

$$\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$$

basis:

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

another basis:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Inner Product

Three definitions

1. Definition based on "external" information. E.g., for ordinary 3-dimensional vectors in space, $\vec{u} \cdot \vec{v} = |u||v| \cos\theta$. Requires external knowledge of lengths $|u|, |v|$ and enclosed angle θ. In complex vector spaces, ordering matters, so we introduce an adjoint vector $|u>^\dagger =: <u|$ to express the ordered inner product as $<u| \cdot |v> =: <u|v>$. Rules: If $<v|w>= c$ then

   require: $<w|v>= c^*$

   $<v|v>$ = real (because $c = c^*$)

   require: $<v|v> \geq 0$; call $\sqrt{<v|v>} = |v|$ "norm" of vector

   require: $<v|v>= 0$ only if $|v>= 0$

   require: $<w|(\alpha|v> + \beta|v>) = \alpha <w|v> + \beta <w|v>$ (linear in the ket)

2. We can use any representation $(a_i), (b_i)$ of the vectors with respect to basis $|i>$:

   $$<u|v>= \sum_i \sum_j a_i^* b_j <i|j>$$

   This becomes most useful if the basis is *orthonormal*: $<i|j>= \delta_{ij}$. In this case, we can represent the adjoint vector $<a|$ with a row $\begin{pmatrix} a_1^* & a_2^* & a_3^* \end{pmatrix}$ and the inner product just becomes the or-

dinary matrix multiplication: $\begin{pmatrix} a_1^* & a_2^* & a_3^* \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = a_1^* b_1 + a_2^* b_2 + a_3^* b_3$

3. Dual Vector Space

   set of bras

   bra: linear operator; a function that turns a vector into a scalar linearly

   $< f| : \mathbb{V} \Rightarrow \mathbb{R}, \mathbb{C}$

   if we know $< f|i >= c_i \forall i \in [1, n]$ then we can apply $< f|$ to any vector in the vector space

   $|u >= \Sigma_{i=1}^{n} \alpha_i |i >$

   $< f|u >= \Sigma_{i=1}^{n} \alpha_i < f|i >= \Sigma_{i=1}^{n} \alpha_i c_i$

   Bra's form a vector space of their own (one can add them and multiply them with scalars); they have the same dimension as the vector space they act on, so one can define a basis for the dual space, as well. In that case, there is a 1-to-1 translation from kets (in the vector space) to bras (in the dual space) simply by using the same coefficients representing them for their respective basis.

Some important relationships

Schwarz Inequality

$| < v|w > | \leq |v| \cdot |w|$

$|z >= |v > - \frac{<w|v>}{|w|^2} |w >$: component of $|v > \perp \text{to} |w >$

Triangle Inequality:

$\big| |v > + |w > \big| \leq |v| + |w|$

Orthonormal basis

Orthogonal: inner product of vectors is 0, despite no vector being 0

Normal: length of each vector is 1

$\Rightarrow$ orthonormal basis

$|i> \forall i \in [1,n]$

$<j|i>= \delta ij$

$|v>= \Sigma_{i=1}^{n} \alpha_i |i>$

$<j|v>=<j|\Sigma_{i=1}^{n} \alpha_i |i>= \alpha_j$

$<u| = \Sigma_{j=1}^{n} \beta_j^* <j|$

$<u|v>= \beta_i^* \alpha_i$

Operators

General case: From one vector space $\mathbb{V}$ to another one, $\mathbb{W}$:

$\Omega : \mathbb{V} \to \mathbb{W}$

Specific example: Adjoint $<v|$ in dual space of $\mathbb{V}$ is a linear operator from $\mathbb{V}$ to the field ($\mathbb{R}$ or $\mathbb{C}$).

Other important case: "square" matrix = operator from $\mathbb{V}$ to $\mathbb{V}$:

$\Omega : \mathbb{V} \to \mathbb{V}$

$<j|\Omega|i>$ is all you need to know (for basis states)

Ω: square matrix of n dimensions

$O_{mn} =< m|\Omega|n>$

Adjoint Operators

$\Omega|v>= |v'>$

$<v'| =< v|\Omega^\dagger$

$\Omega_{mn}^\dagger = \Omega_{nm}^*$

Hermitian operator: $\Omega = \Omega^\dagger$

Unitary operator: $\Omega\Omega^\dagger = \mathbf{1}$

Sum and Product of vector spaces

Given two vector spaces $\mathbb{V}$ and $\mathbb{W}$, we can define the sum

$$\mathbb{V} \oplus \mathbb{W}$$

as the space that contains *all* vectors from $\mathbb{V}$ and *all* vectors from $\mathbb{W}$ as well as all possible sums of such vectors. Its basis is simply the joint set of all basis vectors $\{v_1, v_2, ..., v_n, w_1, w_2, ..., w_m\}$ from $\mathbb{V}$ and from $\mathbb{W}$. Its dimension $n+m$ is the sum of the dimensions of $\mathbb{V}$ and $\mathbb{W}$. A vector in $\mathbb{V} \oplus \mathbb{W}$ is defined by its components ("projection") in both $\mathbb{V}$ and $\mathbb{W}$ and can be represented by a single column that contains its coefficients for the first vector space followed by the ones for the second one.

Given two vector spaces $\mathbb{V}$ and $\mathbb{W}$, we can also define the product

$$\mathbb{V} \otimes \mathbb{W}$$

as the space that contains *all* possible *combinations* $|v> \otimes |w>$ of any vector from $\mathbb{V}$ with any vector from $\mathbb{W}$, as well as all possible sums of such combinations. Its basis is the set of all possible "products" of basis vectors

$$\{v_1 \otimes w_1, v_1 \otimes w_2, ..., v_1 \otimes w_m, v_2 \otimes w_1, ..., v_2 \otimes w_m, ..., v_n \otimes w_1, ..., v_n \otimes w_m\}$$

from $\mathbb{V}$ and $\mathbb{W}$. Its dimension $n \cdot m$ is the product of the dimensions of $\mathbb{V}$ and $\mathbb{W}$. A vector in $\mathbb{V} \otimes \mathbb{W}$ is defined by its $n \cdot m$ coefficients with respect to this product basis. **If** it can be written as product of two vectors from $\mathbb{V}$ and $\mathbb{W}$ it can be represented by a single column that contains m times its first coefficient for $\mathbb{V}$, each time multiplied with one of the coefficients from $\mathbb{W}$, followed by the m times the second coefficient from $\mathbb{V}$ multiplied with the same ones from $\mathbb{W}$ and so on (see example below). However, it is very important to realize that **not** all vectors in $\mathbb{V} \otimes \mathbb{W}$ can be written as simple products of vectors from $\mathbb{V}$ with $\mathbb{W}$. Example: Let $\mathbb{V} = \mathbb{C}^2$ with typical (column!) vector $|v>= (a, b)$ and $\mathbb{W} = \mathbb{C}^3$ with typical column vector $|w>= (x, y, z)$. Then $|v> \otimes |w>= (ax, ay, az, bx, by, bz)$ (again, all written as column - I'm just saving space here). On the other hand, the vector $(0, 0, 1, 1, 0, 0)$ is a perfectly valid vector in $\mathbb{V} \otimes \mathbb{W}$ but cannot be written as product of just two vectors, one each from $\mathbb{V}$ and from $\mathbb{W}$.