0.1 Maxwell equations

It is an experimental fact that the force acting on a point charge due to the electromagnetic field is given (in Gaussian units) by:

$$\vec{F} = m\vec{a} = q\left(\vec{E} + \frac{\vec{v}}{c} \times \vec{B}\right) \tag{1}$$

In order to solve this equation it is necessary to know the vector fields $\vec{E}, \vec{B}$. They are given from the Maxwell equations (in Gauss units):

$$\begin{array}{ll} \vec{\nabla} \cdot \vec{E} = 4\pi\rho & \vec{\nabla} \cdot \vec{B} = 0 \\ \vec{\nabla} \times \vec{B} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} & \vec{\nabla} \times \vec{B} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial \vec{E}}{\partial t} \end{array} \tag{2}$$

where the charge density ρ and the current density J are defined in this way

$$\rho \quad = \quad \frac{\text{amount of charge}}{\text{volume}} \tag{3}$$

$$\vec{J} \cdot d\vec{A} \quad = \quad \frac{dq(d\vec{A})}{dt} \tag{4}$$

($dq(d\vec{A})$ is the amound of charge passing through a small area $d\vec{A}$ during the time dt). These quantities have to fullfill the continuity equation:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}\vec{J} = 0 \tag{5}$$

In fact, this follows from Maxwell's equations: taking the divergence of the fourth equation and comparing to the time derivative of the first one we have:

$$\begin{aligned} \vec{\nabla}\vec{J} &= \frac{c}{4\pi}\vec{\nabla}\left(\vec{\nabla} \times \vec{B} - \frac{1}{c}\frac{\partial E}{\partial t}\right) \\ &= -\frac{1}{4\pi}\frac{\vec{\nabla}\partial\vec{E}}{\partial t} = -\frac{\partial \rho}{\partial t} \end{aligned} \tag{6}$$

For the static case, we can introduce the potentials $\Phi, \vec{A}$ in this way:

$$\begin{aligned} \vec{E} &= -\vec{\nabla}\Phi \\ \vec{B} &= \vec{\nabla} \times \vec{A} \end{aligned} \tag{7}$$

and they satisfy the static Maxwell equations. In the time dependent case we can express $\vec{E}$, $\vec{B}$ in this way:

$$\left\{ \begin{array}{l} \vec{E} = -\vec{\nabla}\Phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t} \\ \vec{B} = \vec{\nabla} \times \vec{A} \end{array} \right. \tag{8}$$

The potentials $\Phi, \vec{A}$ are not uniquely determined. We can define gauge transformations (for arbitrary scalar function $\Lambda(\vec{r}, t)$):

$$\left\{ \begin{array}{l} \vec{A} \to \vec{A}' = \vec{A} + \vec{\nabla}\Lambda \\ \Phi \to \Phi' = \Phi - \frac{\partial \Lambda}{\partial t} \end{array} \right.$$

A change in the potentials $\vec{A}$ and Φ of this type does not induce a change in the fields $\vec{E}$ and $\vec{B}$; in fact:

$$\begin{aligned} \vec{E} &= -\vec{\nabla}\Phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t} = -\vec{\nabla}\Phi' - \frac{1}{c}\frac{\partial \vec{A}'}{\partial t} \\ \vec{B} &= \vec{\nabla} \times \vec{A} = \vec{\nabla} \times \vec{A}'. \end{aligned} \tag{9}$$

This is due to the fact that in classical electrodynamic theory the measurable quantities are only the fields $\vec{E}$ and $\vec{B}$. Putting the expression of the fields $\vec{E}$ and $\vec{B}$ into the maxwell equations we obtain the equation for the vector potential $\vec{A}$ and Φ:

$$\begin{cases} -\vec{\nabla}^2\Phi - \frac{1}{c}\frac{\vec{\nabla}\vec{A}}{\partial t} = 4\pi\rho \\ \vec{\nabla} \times (\vec{\nabla} \times \vec{A}) - \frac{1}{c}\frac{\partial(\vec{\nabla}\Phi)}{\partial t} + \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} = \frac{4\pi}{c}\vec{J} \end{cases} \tag{10}$$

Since we have a certain freedom to change $\vec{A}$, Φ we can choose an appropriate Λ in order to get $\vec{A}$ and Φ in relatively simple form.

Coulomb Gauge $\Phi = 0 \quad \vec{\nabla} \cdot \vec{A} = 0$ the equation for the potentials becomes (only for $\rho = 0$):

$$-\vec{\nabla}^2\vec{A} + \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} = \frac{4\pi}{c}\vec{J} \tag{11}$$

Lorentz Gauge $\frac{1}{c}\frac{\partial \Phi}{\partial t} + \vec{\nabla}\vec{A} = 0$ the equations are:

$$\begin{cases} -\vec{\nabla}^2\Phi + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\Phi = 4\pi\rho \\ -\vec{\nabla}^2\vec{A} + \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} = \frac{4\pi}{c}\vec{J} \end{cases}$$

As an example, if we have $\vec{B}$ we can calculate the corresponding vector potential using the Stokes theorem:

$$\int_{\partial M} \vec{B} \cdot d\vec{a} = \oint_C \vec{A} \cdot d\vec{l}, \tag{12}$$

where ∂M is the surface and C is the contour that bounded the surface.

Example: Outside a solenoid of radius R (aligned with the z-axis) we have

$$\int \vec{B} d\vec{a} = \pi R^2 B = \oint \vec{A} d\vec{l} = 2\pi r A_\phi \to A_\phi = \frac{BR^2}{2r} \tag{13}$$

while inside we have $A_\phi = \frac{r}{2}B$.

0.2 Lagrangian formulation

We return now to our starting point :

$$\vec{F} = m\vec{a} = q\left(\vec{E} + \frac{\vec{v}}{c} \times \vec{B}\right). \tag{14}$$

and we try to find a Lagrangian that gives these equation of motion. In the electrostatic case, without magnetic field, since the energy of a particle is given by:

$$V(\vec{r}) = q\Phi(\vec{r})$$

it is reasonable to use this Lagrangian:

$$\mathcal{L} = \frac{1}{2}m\vec{v}^{\,2} - q\Phi. \tag{15}$$

With the electromagnetic field we have to write a Lagrangian that have to take into account the fact that the field $\vec{B}$ does not do work. So if we write the Lagrangian:

$$\mathcal{L} = \frac{1}{2}m\vec{v}^2 - q\Phi + q\frac{\vec{v}}{c}\cdot\vec{A} \tag{16}$$

and we apply the Euler Lagrange equation of motion we have:

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial v_i}\right) - \frac{\partial\mathcal{L}}{\partial r_i} &= 0 \\ m\dot{v}_i + q\frac{\partial\phi}{\partial r_i} + \frac{d}{dt}\left(\frac{q}{c}A_i\right) - \frac{q}{c}\vec{v}\frac{\partial\vec{A}}{\partial r_i} &= 0 \end{aligned} \tag{17}$$

We note that

$$\frac{dA_i}{dt} = \frac{\partial A_i}{\partial t} + \left(\vec{\nabla}A_i\right)\dot{v}_i, \tag{18}$$

Using the fact that

$$\vec{E} = -\vec{\nabla}\phi - \frac{1}{c}\frac{\partial A}{\partial t} \quad \vec{B} = \vec{\nabla}\times\vec{A}$$

we obtain the equation of motion:

$$m\dot{\vec{v}} = q\vec{E} + \frac{q}{c}\vec{v}\times\vec{B}. \tag{19}$$

So the Lagrangian that we have used reproduces the correct equation of motion. If we want to find the correspondent Hamiltonian we have to perform a Legendre transformation. In order to do this we need the canonical momentum p_i from the Lagrangian:

$$p_i = \frac{\partial\mathcal{L}}{\partial v_i} = mv_i + \frac{q}{c}A_i. \tag{20}$$

It is worth to note that in this case the canonical momentum p_i is different from the ordinary momentum mv_i due to the presence of $\frac{q}{c}A_i$. So the hamiltonian is :

$$H(r,p) = \frac{\vec{p}-\frac{q}{c}\vec{A}}{m}\vec{p} - \frac{m}{2}\left(\frac{\vec{p}-\frac{q}{c}\vec{A}}{m}\right)^2 + q\phi - \frac{q}{c}\left(\frac{\vec{p}-\frac{q}{c}\vec{A}}{m}\right)\vec{A}, \tag{21}$$

that gives :

$$H(r,p) = \frac{1}{2m}\left(\vec{p}-\frac{q}{c}\vec{A}\right)^2 + q\phi. \tag{22}$$

The corresponding Hamilton equations are.

$$\begin{cases} \frac{dp_i}{dt} = -\frac{\partial H}{\partial r_i} = -q\frac{\partial \phi}{\partial r_i} + \left(\frac{\vec{p}-\frac{q}{c}\vec{A}}{m}\right)\left(\frac{q}{c}\frac{\partial \vec{A}}{\partial r_i}\right) \\ \frac{dr_i}{dt} = \frac{\partial H}{\partial p_i} = \frac{p_i - \frac{q}{c}A_i}{m} \end{cases} \tag{23}$$

From the first equation we can see that since

$$\frac{dp_i}{dt} = m\frac{dv_i}{dt} + \frac{q}{c}\frac{\partial A_i}{\partial t} + \frac{q}{c}\left(\vec{v}\cdot\vec{\nabla}\right)A_i$$

,

putting everything together:

$$m\frac{dv_i}{dt} + \frac{q}{c}\frac{\partial A_i}{\partial t} + \frac{q}{c}\left(\vec{v}\cdot\vec{\nabla}\right)A_i = -q\frac{\partial \phi}{\partial r_i} + \frac{q}{c}\sum_j v_j \frac{\partial A_j}{\partial r_i} \tag{24}$$

noting that:

$$\begin{aligned} &\sum_j \left(-v_j\frac{\partial A_i}{\partial r_j} + v_j\frac{\partial A_j}{\partial r_i}\right) \\ &= \sum_j \left(v_j\left(-\frac{\partial A_i}{\partial r_j} + \frac{\partial A_j}{\partial r_i}\right)\right) = \left[\vec{v}\times\left(\vec{\nabla}\times\vec{A}\right)\right]_i \end{aligned} \tag{25}$$

since:

$$\vec{A}\times\left(\vec{B}\times\vec{C}\right) = \left(\vec{A}\vec{C}\right)\vec{B} - \left(\vec{A}\vec{B}\right)\vec{C}. \tag{26}$$

So finally we have the equation of motion for a particle in a electromagnetic field

$$m\frac{dv_i}{dt} = q\left(-\frac{\partial \phi}{\partial r_i} - \frac{1}{c}\frac{\partial A_i}{\partial t}\right) + \frac{q}{c}\left(\vec{v}\times\vec{B}\right)_i = qE_i + \frac{q}{c}(\vec{v}\times\vec{B})_i \tag{27}$$

An important observation: The Hamiltonian is conserved if the scalar and vector potentials do not depend on time, and it corresponds to the simple sum of kinetic and electrostatic potential energy of the particle in that case. So without any further ado, we can deduce that total mechanical energy is conserved in this case. Furthermore, if we can find any cyclic coordinates (i.e., coordinates that the potentials don't depend on, e.g. ϕ), we automatically get a corresponding conserved quantity, e.g. P_ϕ.