**Infinite Vector Space (Shankar, p. 57)**

Infinite vector space implies infinite dimensions
Example: $f : [0, 2\pi] \to \mathbf{C}$
$\therefore$ requirement: $f(0) = f(2\pi)$ , $f$ continuous
In between, a function can do anything, but we assume it's continuous. Is it a vector space? Do we have a basis for that vector space? No, we must expand our notion of "basis" as the basis would have infinite dimensions and infinitely many vectors to describe it. Let's suppose a basis:

$$|f\rangle = \sum_{k=-\infty}^{\infty} a_k |k\rangle$$
$$f(x) = \sum_{-\infty}^{\infty} a_k k(x)$$

Lets use $k(x) = e^{ikx} = e^{2\pi ikx/L}$ where we substitute a more general interval length L for the $2\pi$ from example. Note: The = sign here has a slightly different meaning than in the finite-dimensional case: For each point $x$ in the interval, the sum on the rhs approaches the lhs arbitrarily close as $|k|$ becomes larger and larger, but the convergence is not necessarily uniform. (E.g., the rhs is always differentiable, while the lhs may not be).

Is this set of vectors a basis? Yes, all $|k\rangle$ are linearly independent (can't make harmonics of frequency $k$ from other harmonics with different frequencies). Are they orthogonal? Define inner product between two vectors $|g\rangle, |f\rangle$ via $\langle g|f\rangle = \int_0^L g^*(x) f(x) dx$. This follows the required rules since it is clearly linear in $|f\rangle$ and $\langle f|f\rangle = \int_0^L |f(x)|^2 dx$ is 0 only if $f(x) = 0$ because we assumed continuous.

What is the inner product between two of our proposed basis vectors?

$$\langle k'|k\rangle = \int_0^L e^{-2\pi ik'x/L} e^{2\pi ikx/L} dx = \int_0^L e^{2\pi i(k-k')x/L} dx$$

If $k \neq k' \Rightarrow$ can integrate and have oscillating function

$$\frac{1}{2\pi i(k-k')/L} e^{2\pi ix(k-k')/L} \,|_0^L = 0. \text{ If } k = k' \text{ then } \int_0^L e^0 dx = L$$

Still not normalized yet, $k(x) = \frac{1}{\sqrt{L}} e^{-2\pi ikx/L}$ now, normalized and orthonormal basis

$$a_k = \langle k|f\rangle = \int_0^L \frac{1}{\sqrt{L}} e^{-2\pi ikx/L} f(x) dx = \tilde{f}(k)$$

(Fourier transform). Any operator can be defined by matrix representation: $\langle k'|\Omega|k\rangle = \Omega_{k'k}$ Anything that acts linearly on a function is an operator. Example: $\frac{\partial^2}{\partial x^2} f(x)$

$$\frac{1}{L} \int_0^L e^{-2\pi ik'x/L} \frac{\partial^2}{\partial x^2} e^{2\pi ikx/L} dx \text{ (reminder: acting on right ket)}$$
$$= -\left(\frac{2\pi k}{L}\right)^2 \delta_{k'k}$$

$\therefore$ operator not unitary, yes hermitian because all EV are real (diagonal real), also already know eigenstates (namely, our basis!)
Which EVs of the operator are degenerate? All are 2x degenerate for every value of $\pm k$ except the middle ($k = 0$) is 1x degenerate.

What if operator, $\Omega = \frac{\partial}{\partial x}$?
$\Omega_{k'k} = ik2\pi/L\delta_{k'k} \Leftarrow$ still has diagonal form, basis EF of that operator, but NOT hermitian

We can make this operator Hermitian by multiplying by $-i$ or $i$. Ex: $\Omega = -i\frac{\partial}{\partial x}$ [1]

[1] There is a subtle difficulty with both these "hermitian" operators: They aren't strictly operators *on* the vector space itself! This is because the derivative is ill-defined at the edges (O and $L$) and, even if we define them as appropriate limits, they may not have the same value at 0 and $L$, which was the requirement for members of the vector space. This is a subtlety most texts simply skip over but it has some weird consequences...

Taking another operator, $X$. Define by $X|f\rangle = |g\rangle; g(x) = xf(x)$
To prove hermitian, $\langle h|Xf\rangle \stackrel{?}{=} \langle Xh|f\rangle = \langle h|X^\dagger|f\rangle$
$\langle h|Xf\rangle = \int h^*(x)xf(x)dx; \langle Xh|f\rangle = \int (xh(x))^* f(x)dx$
which is the same; therefore, $X$ is hermitian.
What is $X_{k'k} = \frac{1}{L}\int_0^L xe^{ix2\pi(k-k\prime)/L}dx$ ?
When $k = k\prime \rightarrow \frac{L}{2}$
When $k \neq k\prime \rightarrow \frac{L}{2\pi i(k-k\prime)}$

Obviously a very complicated structure with entries for every matrix element. Apparently $|k>$ basis is not well-adapted for this operator. (I don't even know how to solve for the eigenvalues and eigenfunctions, given that the dimension of the matrix is infinite). Can't we have another, more appropriate basis, i.e. $|x>$? Well, yes and no - it depends on how loosely we want to define the word "basis". Let's take a little detour:

Looking for an operator that can take vectors and turn them into numbers (a "distribution"), i.e. a member of the dual vector space:

$$\Lambda : |f\rangle \rightarrow c.$$

*One* such operator takes function from vector space and maps it onto its value at point x: $\langle x| : |f\rangle \rightarrow f(x)$
Basis of dual vector space!
$\langle x|$, $0 \leq x \leq L$ "is" a basis of dual vector space in the following sense:

$$\Lambda = \langle g| = \int_0^L dxg^*(x)\langle x| \text{ for any arbitrary dual vector } \Lambda =< g|.$$

To prove, apply to some function $f(x)$ (vector $|f>$ from the original vector space):
$\Lambda|f\rangle = \langle g|f\rangle = \int_0^L g^*(x)f(x)dx$ which is the most general form of a linear function from vector space to complex numbers. For any function $|g>$ in the original vector space, we can "convert" it to a dual vector $< g|$ following this recipe. In particular, there is a basis in the dual vector space that corresponds to our $|k>$ basis in the original space: $< k|$ which simply gives the Fourier transform at "frequency" $k$ (a c-number) when applied to a function from the vector space. We can re-interpret: $k(x) = \frac{1}{\sqrt{L}}e^{ik2\pi x/L} = \langle x|k\rangle$

Shouldn't there be also a basis $|x\rangle$ in the original vector space corresponding to the dual space basis $< x|$, such that any vector $|f\rangle$ can be written as $\int_0^L f(x)|x\rangle dx$?The problem with this is that we don't know what $|x>$ is. Can't be a function!

Let's try: $\langle x'|f\rangle = \int_0^L \langle x'|f(x)|x\rangle dx =$
$\int_0^L f(x)\langle x'|x\rangle dx$
should be $= f(x')$
$\therefore$ we must conclude that $\langle x'|x\rangle = \delta(x - x')$ or $\delta(x' - x)$

If $|x\rangle$ were a member of the vector space, then $\langle x'|x\rangle$ should be a function evaluated at $x'$. Due to it being a delta-"function" (really, a distribution), we know its value at every point except at $x' = x$, and it's not continuous. $\therefore$ it cannot be a member of vector space. So now we call $|x\rangle$ a pseudobasis. It's not actually a member of actual vector space, it's not really a function, certainly not continuous, and not normalized. We normalize by getting the inner dot product, but it cannot be normalized since $\langle x|x\rangle = \infty$. However, the members of this pseudo basis *are* orthogonal.

If we can write, for some operator $\Omega$, $\langle k'|\Omega|k\rangle = \Omega_{k'k}$ then $\Omega = \int dkdk'|k'\rangle\Omega_{k'k}\langle k|$.
Back to the $X$ operator:

$$X = \int_0^L |x\rangle x\langle x|dx$$

It is clearly diagonal, so the new pseudo basis forms its eigenvectors with (non degenerate) eigenvalues $x$. Unfortunately, it doesn't have infinitely many but countable matrix elements ($dim = \aleph_0$, as in our previous examples using the $|k>$ basis), but a continuous set of such ($dim = \aleph$), one element (on the diagonal) for every $x \in [0, L]$. This is the price to pay for using a non-normalizable, $\aleph$-dimensional pseudo basis like $|x>$.

So far, we have restricted ourselves to continuous complex functions defined on a finite real integral $f : [0...L] \to \mathbb{C}$.
So let's expand to functions defined on all the real numbers: $f : \mathbb{R} \to \mathbb{C}$. Our definitions of addition of vectors and multiplication by scalars work as before - so we have a vector space. The inner product between two vectors $|g>, |f>$ is now defined via

$$\int_{-\infty}^{\infty} g^*(x) f(x) dx = \langle g|f\rangle$$

How do we gurantee that this exists/produces a finite number? $\Rightarrow$ we will simply make this part of the condition:

$$\langle f|f\rangle < \infty \Longleftarrow \text{HILBERT SPACE}$$

What happens if we go to this Hilbert Space? Our $|k>$ basis is still OK, but $k$ no longer has to be integer: $k(x) = \frac{1}{\sqrt{2\pi}} e^{ikx}$ for any real number $k$. This means that we now have continuous dimension $\aleph$ for this basis as well, and it is no longer normalizable either (i.e., a pseudo basis). So now we have two different "basis sets" with very similar properties:

$$\langle k'|k\rangle = \delta(k' - k)$$
$$\langle x'|x\rangle = \delta(x' - x)$$
$$\langle x|k\rangle = k(x) = \frac{1}{\sqrt{2\pi}} e^{ikx}$$
$$\langle k|x\rangle = k^*(x) = \frac{1}{\sqrt{2\pi}} e^{-ikx}$$

The derivative operators (e.g., $-i\partial/\partial x$) are still diagonal in the $|k>$ basis, and the $X$ operator is still diagonal in the $|x>$ basis. All operators now have continuous ($\aleph$ many) matrix elements, as expressed by either basis. No proper basis exists for this vector space, unfortunately.