PHYS621 - Graduate Quantum Mechanics I

Class Notes

Reminder:

$\hat{X} = \frac{X}{X_0}$; $where\ X_0 = \sqrt{\hbar/_{m\omega}}$

$\hat{P} = \frac{P}{P_0}$; $where\ P_0 = \sqrt{\hbar m\omega}$

Solutions for the harmonic oscillator $\hat{H}|n\rangle = (n+\frac{1}{2})|n\rangle$

$$< \hat{x}|n> = \sqrt{\frac{1}{\sqrt{\pi}2^n n!}} H_n(\hat{x}) e^{-\frac{\hat{x}^2}{2}}$$

Define an operator (normalized)

$$\frac{1}{\sqrt{2}}(\hat{X} + i\hat{P}) = \boldsymbol{a}$$

Why do we need this? Wait and see… this is a very common trick in QM

Operating on an eigenvector of the Hamiltonian, $|n>$ gives,

$$\boldsymbol{a}|n> = \frac{1}{\sqrt{2}}(\hat{X}|n> \ +i\hat{P}|n>)$$

- $\boldsymbol{a}$ is a lowering operator

  Lowers the energy eigenstate to the one below it (see later)
- Is not an hermitian operator since,

$$\boldsymbol{a}^\dagger = \frac{1}{\sqrt{2}}(\hat{X} - i\hat{P}) \neq \boldsymbol{a}$$

It will turn out that $\boldsymbol{a}^\dagger$ is the raising or creation operator.

Introducing these operators allows us to write the Hamiltonian in a better way.

$$\hat{H} = \frac{\hat{X}^2}{2} + \frac{\hat{P}^2}{2}$$

$\hat{H} = \boldsymbol{a}\boldsymbol{a}^\dagger + \ (some\ constant)$:

$$\boldsymbol{a}\boldsymbol{a}^\dagger = \frac{1}{2}(\hat{X}^2 - i\hat{X}\hat{P} + i\hat{P}\hat{X} + \hat{P}^2)$$

$$\boldsymbol{a}\boldsymbol{a}^\dagger = \frac{1}{2}(\hat{X}^2 + i[\hat{P},\hat{X}] + \hat{P}^2)$$

Finding $[\hat{P},\hat{X}]$

$$[\hat{P},\hat{X}] = \left[\frac{P}{P_0},\frac{X}{X_0}\right] = \frac{1}{P_0X_0}[P,X] = \frac{1}{\sqrt{\hbar m\omega \times \hbar/_{m\omega}}}(-i\hbar\mathbb{I}) = -i\mathbb{I}$$

Substituting this back,

$$\boldsymbol{a}\boldsymbol{a}^\dagger = \frac{1}{2}(\hat{X}^2 + \hat{P}^2) + \frac{1}{2}\mathbb{I}$$

$$\boldsymbol{a}\boldsymbol{a}^\dagger - \frac{1}{2}\mathbb{I} = \frac{1}{2}(\hat{X}^2 + \hat{P}^2)$$

$$\boldsymbol{a}\boldsymbol{a}^\dagger - \frac{1}{2}\mathbb{I} = \hat{H}$$

Reversing the order of multiplication of the two operators gives the following,

$$\boldsymbol{a}^\dagger\boldsymbol{a} + \frac{1}{2}\mathbb{I} = \hat{H}$$

This gives us two ways of writing the Hamiltonian.

So let's consider what the lowering and raising operator do if we know nothing about the solutions. Assume there is an eigenvector for this Hamiltonian such that,

$$\hat{H}|n> = \mathcal{E}_n|n>$$

What happens if the raising operator is applied?

$$\hat{H}\boldsymbol{a}^\dagger|n> = \boldsymbol{a}^\dagger\boldsymbol{a}\boldsymbol{a}^\dagger|n> + \frac{1}{2}\boldsymbol{a}^\dagger|n>$$

But from earlier we know that,

$$\boldsymbol{a}\boldsymbol{a}^\dagger = \hat{H} + \frac{1}{2}\mathbb{I}$$

Using this in the equation,

$$\hat{H}\boldsymbol{a}^\dagger|n> = \boldsymbol{a}^\dagger\hat{H}|n> + \frac{1}{2}\boldsymbol{a}^\dagger|n> + \frac{1}{2}\boldsymbol{a}^\dagger|n>$$

$$\hat{H}\boldsymbol{a}^\dagger|n> = \boldsymbol{a}^\dagger\mathcal{E}_n|n> + \boldsymbol{a}^\dagger|n>$$

$$\hat{H}\boldsymbol{a}^\dagger|n> = (\mathcal{E}_n + 1)\boldsymbol{a}^\dagger|n>$$

Similarly,

$$\hat{H}\boldsymbol{a}|n> = (\mathcal{E}_n - 1)\boldsymbol{a}|n>$$

You cannot apply the lowering operator infinitely since then you would get negative energies which is forbidden. So there is a ground state. So there should be one eigenvector with

$$\boldsymbol{a}|0> = 0$$

Where $|0>$ correspond to the lowest possible eigenstate which is the ground state.

So how can we find the value of the ground state ?

$$\hat{H}|0> = \boldsymbol{a}^\dagger\boldsymbol{a}|0> + \frac{1}{2}|0>$$

$$\hat{H}|0> = \frac{1}{2}|0>$$

This gives $\mathcal{E}_0 = \frac{1}{2}$

So using this we can write $E_n = \left(n + \frac{1}{2}\right)\hbar\omega$

We know that,

$\boldsymbol{a}^\dagger\,|0> \;=\; c_0\,|1>$
$\boldsymbol{a}^\dagger\,|1> \;=\; c_1\,|2>$
$\vdots \qquad \vdots \qquad \vdots$
$\boldsymbol{a}^\dagger\,|n> \;=\; c_n\,|n+1>$

What we need to know are the coefficients.
Since the vectors are orthonormalized,

$< n|\boldsymbol{a}\,\boldsymbol{a}^\dagger\,|n> \;=\; < n+1|\,{c_n}^*\,c_n\,|n+1> = \;|c_n|^2$

$< n|\,\hat{H} + \frac{1}{2}\,|n> = < n|n+1\,|n> = n+1$

$c_n = \sqrt{n+1}$

Using this result we can write for any arbitrary $|n>$

$$|n> = \frac{1}{\sqrt{n!}}\left(\boldsymbol{a}^\dagger\right)^n|0>$$

If the order of multiplication is reversed,

$< n\big|\,\boldsymbol{a}^\dagger\,\boldsymbol{a}\big|n> = n$

Where $\boldsymbol{a}^\dagger\,\boldsymbol{a}$ is the counting operator, it counts how many quanta of energy you have.

Some applications

Note that $\frac{1}{\sqrt{2}}\left(a + a^\dagger\right) = \hat{X}; \frac{1}{i\sqrt{2}}\left(a - a^\dagger\right) = \hat{P}$

Using this we find:

$$< n+1\big|\hat{X}\big|n> = \sqrt{\frac{n+1}{2}} \qquad and \qquad < n+1\big|\hat{P}\big|n> = i\sqrt{\frac{n+1}{2}}$$

It follows that,

$$< n-1\big|\hat{X}\big|n> = < n|Y|n-1>^* = \sqrt{n/2}$$

$$< n-1\big|\hat{P}\big|n> = < n\big|\hat{P}\big|n-1>^* = -i\sqrt{n/2}$$

So now we can write down,

$$\hat{X}|n> = \sqrt{\frac{n+1}{2}}\ \Big|n+1> + \sqrt{n/2}\,|n-1>$$

$$\hat{P}|n> = i\sqrt{\frac{n+1}{2}}\ \Big|n+1> - i\sqrt{n/2}\,|n-1>$$

To find out everything about the operator $\hat{X}$ we need to know the elements,

$< n|\hat{X}|m >$

From above, we know this for two cases, which is

$n = m \pm 1$

The rest of the elements,

$n = m$, $n > m + 1$ and $n < m - 1$ are zero. (For both $Y\ and\ \hat{P}$)

Now we get a matrix in the following format,

$$\begin{pmatrix} 0 & * & 0 & 0 & 0 & 0 \\ * & 0 & * & 0 & 0 & 0 \\ 0 & * & 0 & * & 0 & 0 \\ 0 & 0 & * & 0 & * & 0 \\ 0 & 0 & 0 & * & 0 & * \\ 0 & 0 & 0 & 0 & * & 0 \end{pmatrix}$$

What is ΔX ?

To find this, first we calculate $(\Delta X\ )^2$

$$(\Delta X\ )^2{}_{|n>} = < X^2 >_{|n>}$$

$$(\Delta X\ )^2{}_{|n>} = < n|X^2|n > = |X|n >|^2 = X_0^2\ \left|\hat{X}|n >\right|^2$$

$$(\Delta X\ )^2{}_{|n>} = X_0^2\ \left|\sqrt{\tfrac{n+1}{2}}|n+1> + \sqrt{n/2}\,|n-1>\right|^2$$

Considering the orthonormality,

$$(\Delta X\ )^2{}_{|n>} = X_0^2\ \left(\frac{n+1}{2} + \frac{n}{2}\right)$$

$$(\Delta X\ )^2{}_{|n>} = \frac{\hbar}{m\omega}\left(n + \frac{1}{2}\right)$$

From this we can calculate Δx

$$\Delta X = \sqrt{\frac{\hbar}{m\omega}\left(n+\frac{1}{2}\right)} = \sqrt{\frac{\hbar\omega}{m\omega^2}\left(n+\frac{1}{2}\right)} = \sqrt{\frac{E_n}{k}} \qquad ;where\ k\ is\ the\ spring\ constant$$

Likewise for $\hat{P}$

$$\Delta\hat{P} = \sqrt{\hbar m\omega\left(n+\frac{1}{2}\right)}$$

So now we can write,

$$\Delta X \Delta \hat{P} = \hbar\left(n + \frac{1}{2}\right)$$

This value is always greater than or equal to $\frac{\hbar}{2}$

- ❖ What is the amplitude of a classical oscillator?

$E = \frac{1}{2} kA^2$ At max

$$A = \sqrt{\frac{2E}{k}}$$

So the uncertainty is of the same order of magnitude as the classical amplitude (but a factor √2 smaller).