Quantum Mechanics
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Addition Of Angular Mometa:

Let us consider two particles. Let 1st particle is ~~explained~~ represented in $\mathbb{V}_1$ space & 2nd particle is ~~explained~~ represented in $\mathbb{V}_2$ space. Then the system of these particles is ~~explained~~ represented in $\mathbb{V}_1 \otimes \mathbb{V}_2$ space.

Let us consider the system of two spin half particles. Since spin half particle is explained in 2 dimensional complex vector space $\mathbb{C}^2$. The system of these particles is spanned by four basis vectors in $\mathbb{C}^2 \otimes \mathbb{C}^2$ space. i.e. $|m_1\rangle \otimes |m_2\rangle$ is the basis for two particle system. And arbitrary state can be expressed as superposition of these four vector in four dimensional space.

$$\left|\tfrac{1}{2}\right\rangle \otimes \left|\tfrac{1}{2}\right\rangle = \begin{pmatrix} 1 \begin{pmatrix}1\\0\end{pmatrix} \\ 0 \begin{pmatrix}1\\0\end{pmatrix} \end{pmatrix} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}$$

$$\left|\tfrac{1}{2}\right\rangle \otimes \left|-\tfrac{1}{2}\right\rangle = \begin{pmatrix} 1 \begin{pmatrix}0\\1\end{pmatrix} \\ 0 \begin{pmatrix}0\\1\end{pmatrix} \end{pmatrix} = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}$$

$$\left|-\tfrac{1}{2}\right\rangle \otimes \left|\tfrac{1}{2}\right\rangle = \begin{pmatrix} 0 \begin{pmatrix}1\\0\end{pmatrix} \\ 1 \begin{pmatrix}1\\0\end{pmatrix} \end{pmatrix} = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}$$

$$\left|-\tfrac{1}{2}\right\rangle \otimes \left|\tfrac{1}{2}\right\rangle = \begin{pmatrix} 0 \begin{pmatrix}0\\1\end{pmatrix} \\ 1 \begin{pmatrix}0\\1\end{pmatrix} \end{pmatrix} = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

Operators:-

Any operators will represented by 4×4 matrix.

eg.

$$S_{1z} = S_z \cdot \mathbb{1} = \frac{\hbar}{2}\begin{pmatrix} 1\cdot\mathbb{I} & 0\cdot\mathbb{I} \\ 0\,\mathbb{I} & -1\cdot\mathbb{I} \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} 1\begin{pmatrix}1&0\\0&1\end{pmatrix} & 0\begin{pmatrix}1&0\\0&1\end{pmatrix} \\ 0\begin{pmatrix}1&0\\0&1\end{pmatrix} & -1\begin{pmatrix}1&0\\0&1\end{pmatrix} \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1 \end{pmatrix}$$

$$S_{2z} = \mathbb{I}\cdot S_z = \begin{pmatrix} 1\cdot S_z & 0\cdot S_z \\ 0\cdot S_z & -1\cdot S_z \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} 1\begin{bmatrix}1&0\\0&-1\end{bmatrix} & 0\begin{bmatrix}1&0\\0&-1\end{bmatrix} \\ 0\begin{bmatrix}1&0\\0&-1\end{bmatrix} & 1\begin{bmatrix}1&0\\0&-1\end{bmatrix} \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} 1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&-1 \end{pmatrix}$$

- $S_{1+} = \hbar\begin{pmatrix} 0 & \mathbb{I} \\ 0 & 0 \end{pmatrix} = \begin{bmatrix} 0&0&1&0\\0&0&0&1\\0&0&0&0\\0&0&0&0 \end{bmatrix}$; $S_{1-} = \hbar\begin{bmatrix} 0 & 0 \\ \mathbb{I} & 0 \end{bmatrix} = \hbar\begin{bmatrix} 0&0&0&0\\0&0&0&0\\1&0&0&0\\0&1&0&0 \end{bmatrix}$

- $S_{2+} = \begin{pmatrix} S_+ & 0 \\ 0 & S_+ \end{pmatrix} = \hbar\begin{pmatrix} 0&1&0&0\\0&0&0&0\\0&0&0&1\\0&0&0&0 \end{pmatrix}$; $S_{2-} = \begin{pmatrix} S_- & 0 \\ 0 & S_- \end{pmatrix} = \hbar\begin{bmatrix} 0&0&0&0\\1&0&0&0\\0&0&0&0\\0&0&1&0 \end{bmatrix}$

Note: $S_1 = S_1^{(1)} \otimes \mathbb{I}^2$ & $S_2 = \mathbb{I}^{(1)} \otimes S_2^{(2)}$

$\circ\ S_1^2 = \frac{3\hbar^2}{4}\begin{bmatrix} \mathbb{I} & 0 \\ 0 & \mathbb{I} \end{bmatrix} = \frac{3\hbar^2}{4}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$S_2^2 = \begin{bmatrix} 1\cdot S^2 & 0 \\ 0 & 1 S^2 \end{bmatrix} = \frac{3\hbar^2}{4}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Note: $[\underbrace{S_{1x}}_{\text{acts on 1st only}}, \overbrace{S_{2y}}^{\text{acts on 2nd only}}] = 0$

<u>Rotation by an angle ϕ around z-axis:</u>

As we know rotation operator on a single spin half is

$$D^{1/2}(R(\theta\hat{n})) = e^{\frac{-i\theta\hat{n}\cdot\vec{S}}{\hbar}}$$

For the system of 2 spin half particle around z-axis, operator is,

$$e^{-i\phi\left(\frac{S_{1z}+S_{2z}}{\hbar}\right)}$$

• $S_{tot\,z} = S_{1z} + S_{2z} = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} + \frac{\hbar}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \hbar\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$

$$S_{tot\,z}\,|m_1\rangle \otimes |m_2\rangle = \underbrace{(m_1+m_2)}_{}\,|m_1\rangle \otimes |m_2\rangle$$

$M = m_1 + m_2$; $M = 0, 1, -1$

$\circ\ S_{tot}^2 = S_{tot-}S_{tot+} + \hbar S_{tot\,z} + S_{tot\,z}^2$

$= \hbar^2\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \hbar^2\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} + \hbar^2\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

$= \hbar^2\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} + \hbar^2\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} + \hbar^2\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

$= \hbar^2\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$

∘ $s(s+1) = 2 \Rightarrow s = 1$ for 2 "corner" elements"

• How about the following state: $|\psi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\0\\0\end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix}0\\0\\1\\0\end{pmatrix}$

Here, $$S^2_{tot}|\psi\rangle = \hbar^2 \begin{bmatrix}2&0&0&0\\0&1&1&0\\0&1&1&0\\0&0&0&2\end{bmatrix}\left\{\underbrace{\frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\0\\0\end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix}0\\0\\1\\0\end{pmatrix}}_{=\frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\1\\0\end{pmatrix}}\right\}$$

$$= 2\hbar^2 \cdot \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\1\\0\end{pmatrix}$$

We write, $\frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\1\\0\end{pmatrix}$ as $|1,0\rangle$ is eigen state of S^2_{tot} $[s(s+1)=2]$

||ly, $\frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\-1\\0\end{pmatrix}$ as $|0,0\rangle$ ⟶ eigenstate of S^2_{tot} w/ EV $s=0$

$\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$ as $|1,1\rangle$

$\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$ as $|1,-1\rangle$

} are also eigen state of S^2_{tot}. w/ e.v. $s=1$

So, the total angular momentum $|S,M\rangle$ representation for two spin half, the four basis states are $|1,1\rangle$, $|1,0\rangle$, $|1,-1\rangle$ & $|0,0\rangle$. Still there are 4 basis vectors, orthogonal to each other are complete.

The three spin-1 states are called triplets & the solitary spin-0 state is called the singlet.

• Since, $\mathbb{C}^2 \otimes \mathbb{C}^2 = \mathbb{C}^4$

$\approx \mathbb{C}^3 \oplus \mathbb{C}^1 = \mathbb{C}^4$

So, the direct product of spin half Hilbert space is a direct sum of spin 1 space & spin 0 space.

• 4x4 operators are reducible representation, it can be reduced to sum of smaller dimensional representation. Irreducible is one which have no subspace to reduced to sum.

$$\begin{array}{ccccc} \frac{1}{2} \otimes & \frac{1}{2} & = & 1 \oplus & 0 \\ \Downarrow & & & \Downarrow & \Downarrow \\ 2 \times & 2 & & 3 & + 1 \end{array}$$

- Interaction Hamiltonian:-

If the spins are mutually interacting

$$\vec{H}_{int} = A\vec{S}_1 \cdot \vec{S}_2$$

where, A is a constant.

- Now we write total-j ket as

$$|J, M\rangle = \sum_{m_1 m_2} \langle j_1 m_1 | j_2 m_2 | JM\rangle |j_1 m_1\rangle \otimes |j_2 m_2\rangle$$

$$= \sum_{m_1, m_2} \underbrace{\langle j_1 m_1 j_2 m_2 | JM\rangle}_{\text{are called Clebsch-Gordan coefficients.}} |j_1 m_1, j_2 m_2\rangle$$

For, $J = S$, $S_1 = S_2 = \frac{1}{2}$

$$\langle \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}\ \tfrac{1}{2} | 1\ 1\rangle = 1$$

$$\langle \tfrac{1}{2}\ \tfrac{1}{2}\ {-\tfrac{1}{2}}\ \tfrac{1}{2} | 1\ 0\rangle = \tfrac{1}{\sqrt{2}}$$

$$\langle \tfrac{1}{2}\ \tfrac{1}{2}\ \tfrac{1}{2}\ {-\tfrac{1}{2}} | 0\ 0\rangle = -\tfrac{1}{\sqrt{2}}$$

$$\langle \tfrac{1}{2}\ {-\tfrac{1}{2}}\ \tfrac{1}{2}\ \tfrac{1}{2} | 0\ 0\rangle = \tfrac{1}{\sqrt{2}}$$

Some rules & properties:-

- $\langle j_1 m_1 j_2 m_2 | JM\rangle = 0$ unless $m_1 + m_2 = M$
- $|j_1 - j_2| \leq J \leq |j_1 + j_2|$
- $\langle j_1 m_1 j_2 m_2\rangle \neq 0$ if $j_1 - j_2 \leq j \leq j_1 + j_2$
- $\langle$ They are real.

- For fixed j, ~~it is in~~ $\mathbb{C}^{2j+1} \otimes \mathbb{C}^{2j+1}$ space can be expressed ~~as~~

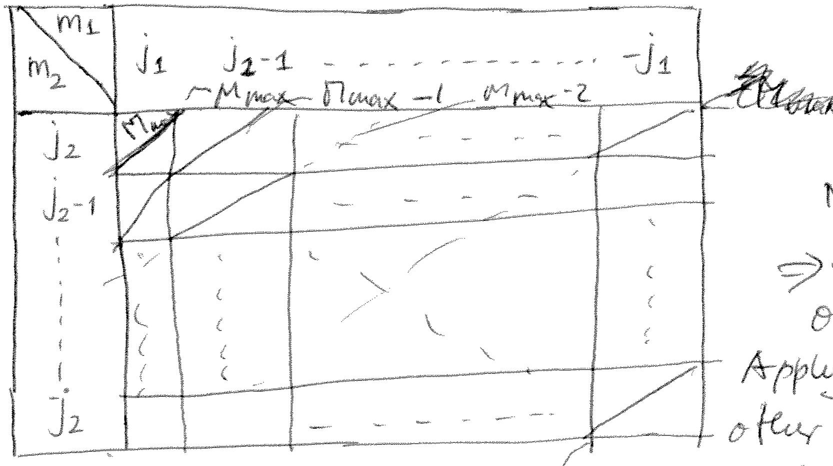
$$\mathbb{C}^{2j_{min}+1} \oplus \mathbb{C}^{2(2j_{min}+1)+1} \cdots\cdots + \mathbb{C}^{2\cdot 2j_{max}+1}$$

where, $j_{min} = j_1 - j_2$; $j_1 \geq j_2$

$j_{max} = j_1 + j_2$

Then, $$(2j_1+1)(2j_2+1) = \sum_{j=j_{min}}^{j=j_{max}} (2j+1) = \underbrace{\frac{(2j_{max}+1) + (2j_{min}+1)}{2}}_{\text{Average}} \underbrace{(2j_2+1)}_{\text{no. of terms}}$$

$$= \frac{2(j_1+j_2) + 2 + 2(j_1-j_2)}{2} (2j_2+1) = (2j_1+1)(2j_2+1)$$



$M_{max} = j_1 + j_2$

$\Rightarrow J = j_1 + j_2$

only one state possible (CG = 1,

Apply S_- $2J+1$ times to get all other M substates down to $-M_{max} = -j_1 - j_2$

$-M_{max} = -j_1 - j_2$

Then start w/ single remaining orthogonal $M_{max}-1$ state $\rightarrow$ must belong to j_1+j_2-1 $\rightarrow$ continue with $2J-1$ times S_- continue with j_1+j_2-2 $\rightarrow$ all the way to j_1-j_2

- Electron in H-atom:

$$\sum_{l=0}^{\infty} \mathbb{V}_{radial}(l) \otimes \underbrace{\mathbb{C}^{2l+1}}_{Y^{lm}(\theta,\phi)}$$

For fixed l, ; $m = -l, \ldots\ldots l$

Since electron have spin,

$$\sum_{l=0}^{\infty} \mathbb{V}_{radial}(l) \otimes \underbrace{\mathbb{C}^{2l+1} \otimes \mathbb{C}^{2}_{spin}}_{\mathbb{C}^{2(l+1/2)+1} \oplus \mathbb{C}^{2(l+1/2)+1}}$$

where, $j = l-s,\ l+s$