The following formulas might be useful for the solution of the exam problems:

FUNDAMENTAL CONSTANTS

ϵ_0	$= 8.85 \times 10^{-12}\,\mathrm{C^2/Nm^2}$	(permittivity of free space)
μ_0	$= 4\pi \times 10^{-7}\,\mathrm{N/A^2}$	(permeability of free space)
c	$= 3.00 \times 10^{8}\,\mathrm{m}/s$	(speed of light)
e	$= 1.60 \times 10^{-19}\,\mathrm{C}$	(charge of the electron)
m	$= 9.11 \times 10^{-31}\,\mathrm{kg}$	(mass of the electron)

SPHERICAL AND CYLINDRICAL COORDINATES

Spherical

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases} \qquad \begin{cases} \hat{\mathbf{x}} = \sin\theta\cos\phi\,\hat{\mathbf{r}} + \cos\theta\cos\phi\,\hat{\boldsymbol{\theta}} - \sin\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} = \sin\theta\sin\phi\,\hat{\mathbf{r}} + \cos\theta\sin\phi\,\hat{\boldsymbol{\theta}} + \cos\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} = \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}} \end{cases}$$

$$\begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \theta = \tan^{-1}(\sqrt{x^2+y^2}/z) \\ \phi = \tan^{-1}(y/x) \end{cases} \qquad \begin{cases} \hat{\mathbf{r}} = \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol{\theta}} = \cos\theta\cos\phi\,\hat{\mathbf{x}} + \cos\theta\sin\phi\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \end{cases}$$

Cylindrical

$$\begin{cases} x = s\cos\phi \\ y = s\sin\phi \\ z = z \end{cases} \qquad \begin{cases} \hat{\mathbf{x}} = \cos\phi\,\hat{\mathbf{s}} - \sin\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} = \sin\phi\,\hat{\mathbf{s}} + \cos\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} = \hat{\mathbf{z}} \end{cases}$$

$$\begin{cases} s = \sqrt{x^2+y^2} \\ \phi = \tan^{-1}(y/x) \\ z = z \end{cases} \qquad \begin{cases} \hat{\mathbf{s}} = \cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}} \\ \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \\ \hat{\mathbf{z}} = \hat{\mathbf{z}} \end{cases}$$

VECTOR IDENTITIES

TRIPLE PRODUCTS

(1) $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$

(2) $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$

PRODUCT RULES

(3) $\nabla(fg) = f(\nabla g) + g(\nabla f)$

(4) $\nabla(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A}$

(5) $\nabla \cdot (f\mathbf{A}) = f(\nabla \cdot \mathbf{A}) + \mathbf{A} \cdot (\nabla f)$

(6) $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$

(7) $\nabla \times (f\mathbf{A}) = f(\nabla \times \mathbf{A}) - \mathbf{A} \times (\nabla f)$

(8) $\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A})$

SECOND DERIVATIVES

(9) $\nabla \cdot (\nabla \times \mathbf{A}) = 0$

(10) $\nabla \times (\nabla f) = 0$

(11) $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$

FUNDAMENTAL THEOREMS

Gradient Theorem: $\displaystyle\int_a^b (\nabla f) \cdot d\mathbf{l} = f(b) - f(a)$

Divergence Theorem: $\displaystyle\int_{\text{volume}} (\nabla \cdot \mathbf{A})\, d\tau = \oint_{\text{surface}} \mathbf{A} \cdot d\mathbf{a}$

Curl Theorem: $\displaystyle\int_{\text{surface}} (\nabla \times \mathbf{A}) \cdot d\mathbf{a} = \oint_{\text{line}} \mathbf{A} \cdot d\mathbf{l}$

VECTOR DERIVATIVES

Cartesian. $d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}; \quad d\tau = dx\,dy\,dz$

Gradient : $$\nabla t = \frac{\partial t}{\partial x}\hat{\mathbf{x}} + \frac{\partial t}{\partial y}\hat{\mathbf{y}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$

Divergence : $$\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

Curl : $$\nabla \times \mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}}$$

Laplacian : $$\nabla^2 t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}$$

Spherical. $d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}; \quad d\tau = r^2\sin\theta\,dr\,d\theta\,d\phi \quad = r^2 dr\, d\cos\theta\, d\phi$

Gradient : $$\nabla t = \frac{\partial t}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial t}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}}$$

Divergence : $$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}$$

Curl : $$\nabla \times \mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial\phi}\right]\hat{\mathbf{r}}$$
$$+ \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial\theta}\right]\hat{\boldsymbol{\phi}}$$

Laplacian : $$\nabla^2 t = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial t}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 t}{\partial\phi^2}$$

Cylindrical. $d\mathbf{l} = ds\,\hat{\mathbf{s}} + s\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}; \quad d\tau = s\,ds\,d\phi\,dz$

Gradient : $$\nabla t = \frac{\partial t}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial t}{\partial\phi}\hat{\boldsymbol{\phi}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$

Divergence : $$\nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial\phi} + \frac{\partial v_z}{\partial z}$$

Curl : $$\nabla \times \mathbf{v} = \left[\frac{1}{s}\frac{\partial v_z}{\partial\phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{\mathbf{s}} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right]\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial\phi}\right]\hat{\mathbf{z}}$$

Laplacian : $$\nabla^2 t = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial t}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 t}{\partial\phi^2} + \frac{\partial^2 t}{\partial z^2}$$

Maxwell's Equations

Translation of equations in SI system to those using Gauß system:

$q, \rho, I, \ldots$ [Gauß] $= \frac{1}{\sqrt{4\pi\varepsilon_0}} q, \rho, I, \ldots$ [SI]

$\vec{E}$ [Gauß] $= \sqrt{4\pi\varepsilon_0}\vec{E}$ [SI]; $\vec{B}$ [Gauß] $= \sqrt{4\pi\varepsilon_0}c\vec{B}$ [SI]

Maxwell's Equations in free space:

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0} \text{ [SI]}, \vec{\nabla}\cdot\vec{E} = 4\pi\rho \text{ [Gauß]}; \vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \text{ [SI]}, \vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t} \text{ [Gauß]}$$

$$\vec{\nabla}\cdot\vec{B} = 0 \quad ; \quad \vec{\nabla}\times\vec{B} = \mu_0\left(\vec{j} + \varepsilon_0\frac{\partial\vec{E}}{\partial t}\right) \text{ [SI]}, \vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{j} + \frac{1}{c}\frac{\partial\vec{E}}{\partial t} \text{ [Gauß]}$$

Lorentz Force Law: $\vec{F} = Q\left(\vec{E} + \vec{v}\times\vec{B}\right)$ [SI], $\vec{F} = Q\left(\vec{E} + \frac{\vec{v}}{c}\times\vec{B}\right)$ [Gauß];

Continuity Equation: $\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0$

AUXILLARY FIELDS

Definitions:

$$\begin{cases} \mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P} \\ \mathbf{H} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M} \end{cases}$$

In linear media:

$$\begin{cases} \mathbf{P} = \epsilon_0\chi_e\mathbf{E}, & \mathbf{D} = \epsilon\mathbf{E} \\ \mathbf{M} = \chi_m\mathbf{H}, & \mathbf{H} = \frac{1}{\mu}\mathbf{B} \end{cases}$$

Potential Formulation of Maxwell's Equations [SI]

$\vec{E} = -\vec{\nabla}V - \dot{\vec{A}} \qquad \vec{B} = \vec{\nabla}\times\vec{A}$

Coulomb Gauge: $\vec{\nabla}\cdot\vec{A} = 0 \quad \Rightarrow$

$\vec{\nabla}^2 V = -\frac{\rho}{\varepsilon_0}$; $\Box\vec{A} = \mu_0\vec{J} - \frac{1}{c^2}\frac{\partial}{\partial t}\vec{\nabla}V$ where

$$\Box = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2 = \varepsilon_0\mu_0\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2$$

Lorentz Gauge: $\frac{1}{c^2}\frac{\partial}{\partial t}V + \vec{\nabla}\cdot\vec{A} = 0 \quad \Rightarrow$

$\Box V = \frac{\rho}{\varepsilon_0}$; $\Box\vec{A} = \mu_0\vec{J}$

Lorentz Force Law: $\frac{d}{dt}\left(\vec{p} + q\vec{A}\right) = q\vec{\nabla}\left[\vec{v}\cdot\vec{A}(\vec{r}) - V(\vec{r})\right]$

ENERGY, MOMENTUM, AND POWER

Energy: $$W = \frac{1}{2}\int\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right)d\tau$$

Momentum: $$\mathbf{P} = \epsilon_0\int(\mathbf{E}\times\mathbf{B})\,d\tau$$

Poynting vector: $$\mathbf{S} = \frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B})$$

Larmor formula: $$P = \frac{1}{4\pi\epsilon_0}\frac{2}{3}\frac{q^2a^2}{c^3}$$

<u>The Dirac Delta Function:</u>

In one dimension: $\delta(x-x') = \begin{cases} 0 \text{ for } x \neq x' \\ \infty \text{ for } x = x' \end{cases}$

$$\int_a^b f(x)\delta(x-x_o)\,dx = \begin{cases} 0 \text{ for } x_o \notin [a,b] \\ f(x_o) \text{ for } x_o \in [a,b] \end{cases}$$

In three dimensions: $\delta^3(\vec{r}-\vec{r}_o) = \delta(x)\delta(y)\delta(z)$

For some smooth scalar field $f(\mathbf{r})$: $\iiint_{\text{Volume}} f(\vec{r}\,')\delta^3(\vec{r}-\vec{r}\,')d^3r' = f(\vec{r})$ if $\vec{\mathrm{r}} \in$ Volume

If f(x) is a strictly monotonous function with one zero x_0 ($f(x_0) = 0$), then

$$\delta(\,f(x)\,) = \frac{1}{f'(x_0)}\cdot\delta(x - x_0)$$

Divergence of Coulomb field: $\vec{\nabla}_r\cdot\frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3} = -\vec{\nabla}_r^{\,2}\frac{1}{|\vec{r}-\vec{r}\,'|} = 4\pi\delta^3(\vec{r}-\vec{r}\,')$
