Graduate Quantum Mechanics II - Problem Set 1

Problem 1)

Show, without any complicated math, that the trace of the density matrix must always be equal to 1 (just from its properties with respect to expectation values of observables).

Problem 2)

Assume we have a large ensemble of spin-1/2 particles (non-interacting) which can be described by the density matrix ρ (or, equivalently, the polarization $\vec{P}$).

a) Explain why for any such ensemble of spin-1/2 particles, $\left|\vec{P}\right| \leq 1$, and why $\left|\vec{P}\right| = 1$ only for a "pure ensemble".

b) Show that for this same situation, trace(ρ^2) ≤ 1, and trace(ρ^2) =1 only for a pure ensemble. Also show that trace(ρ) =1 always.

c) Show that, for any observable **O**, we can write $\left\langle \bar{\mathbf{O}} \right\rangle = a_0\left(1+\vec{A}\cdot\vec{P}\right)$ (average over the whole ensemble), where $\vec{A}=\left(A_x, A_{y,}, A_{z,}\right)$ are three real constants depending only on **O**.

Problem 3)

We send a large ensemble of spin-1/2 particles through a Stern-Gerlach apparatus oriented to select the z-component of the magnetic moment. We find that roughly ½ of the ensemble ends up on the upper trajectory (corresponding to s_z "up") and the other ½ on the lower trajectory (s_z down). Which of the following statements can you make with confidence? Explain your reasoning in each case!

i) Before entering the Stern-Gerlach apparatus, the ensemble was completely unpolarized.

ii) We can not make any conclusion about the initial distribution of spin states in the ensemble

iii) After the measurement, we have two separate ensembles with perfectly known state vectors (two pure ensembles)

iv) If we mix those two ensembles after the measurement thoroughly (still no interaction), we will get a completely unpolarized ensemble.

See reverse side

Problem 4)

We prepare a pure ensemble of 10,000 (spinless) particles, each in the same state, namely a "Gaussian wave packet" with centroid p_0 and width Δ_p.We send this ensemble through a magnet and then measure how many of them end up at each of 10 detectors, which correspond to 10 different possible trajectories through the magnet, with 10 specific momenta. The probability for each individual particle to have the right momentum to end up in detector i turns out to be $0.01 \cdot e^{-\frac{(i-5)^2}{2}}$. Answer the following questions:

i) How many particles do you expect to end up in detector i=3?

ii) What is the probability that you observe exactly 12 particles from the ensemble ending up in detector i=3?

iii) Explain what you assumed for the underlying probability distribution to answer ii) and why you are justified in your assumption.