Graduate Quantum II
Identical Particles
Spring 2013

1 Identical Particles

Before, we discussed entanglement where two particles were created in a well-defined total spin state. Now we need to address particles that were always entangled (how created doesn't matter). When we talk about identical particles, we mean that they share all the same intrinsic quantum numbers-not necessarily where they are or what direction, but we mean what they are. Because it is impossible to distinguish one from the other, we must write them as a linear combination of the two possible orderings:

$$\alpha|n_a, m_a \dots n_b, m_b \dots\rangle + \beta|n_b, m_b \dots n_a, m_a \dots\rangle$$

where the quantum numbers $n_a, m_a, ...$ etc. completely determine the state of one particle. So if I choose to interchange the particles, then I can just switch the coefficients α and β:

$$\beta|n_a, m_a \dots n_b, m_b \dots\rangle + \alpha|n_b, m_b \dots n_a, m_a \dots\rangle$$

Because this exchange does not change the physical state of the system in any detectable manner (given the two particles are indistinguishable), the new state vector must equal the original multiplied by some complex number, c:

$$\begin{aligned} &\beta|n_a, m_a \dots n_b, m_b \dots\rangle + \alpha|n_b, m_b \dots n_a, m_a \dots\rangle \\ &= c\left(\alpha|n_a, m_a \dots n_b, m_b \dots\rangle + \beta|n_b, m_b \dots n_a, m_a \dots\rangle]\right) \end{aligned}$$

Now, looking at the relationships between the coefficients:

$$\begin{aligned} \beta &= c\alpha \\ \alpha = c\beta \rightarrow \alpha &= c^2\alpha \\ \rightarrow c &= \pm 1 \end{aligned}$$

Now we have two cases that result when we exchange particles. The first case, we have $c = 1$, $\alpha = \beta$:

$$|\Psi_s\rangle = |n_a, m_a \dots n_b, m_b \dots\rangle + |n_b, m_b \dots n_a, m_a \dots\rangle$$

In the second case, we have $c = -1$, $\alpha = -\beta$:

$$|\Psi_a\rangle = |n_a, m_a \dots n_b, m_b \dots\rangle - |n_b, m_b \dots n_a, m_a \dots\rangle$$

2 Fermions and Bosons

Two different types of particles:
Bosons: $|\Psi\rangle = |\Psi_s\rangle$, symmetric wave functions
Fermions: $|\Psi\rangle = |\Psi_a\rangle$, anti-symmetric wave functions

How do we know which type we have? Anti-symmetric makes it impossible to have exactly the same quantum numbers or we get zero. This is the Pauli-Exclusion Principle which states that we can't be in the same quantum state if we specify all the quantum numbers of a state.

Fermions try to remain apart whereas bosons try try to remain together. Fermion are matter/material, and matter can never occupy exactly the same space. Bosons are responsible for the interactions between fermions; they are carriers of interaction. Having many bosons together creates stronger forces. For example, one can think of the laser.

Bosons: H(Higgs), γ, W, Z, G(gluons), g(gravitons) as well as composites like ^{4}He, π^0...
Fermions: e^-, μ^-, τ^-, ν_e, ν_μ, ν_τ (leptons), u, c, t, d, s, b (quarks) and composites: proton, neutron, ^{3}He,... (most if not all of these fermions have anti-particles, doubling the total).

Bosons have whole integer spins while Fermions have half-integer spins.

3 Technicalities of the Vector Space

2 spin 0 Bosons: Each boson lives in its on space, **V**:
Two-particle space in general: $\mathbf{V} \otimes \mathbf{V}$ with basis $|b_i\rangle \otimes |b_j\rangle$ for all possible combinations of single-particle basis vectors in **V**.

We introduce a new basis:

$$|b_i, b_j, S\rangle = \frac{1}{\sqrt{2}}(|b_i\rangle \otimes |b_j\rangle + |b_j\rangle \otimes |b_i\rangle)$$

for

$$i < j, i \neq j$$

and

$$|b_i, b_i, S\rangle = |b_i\rangle \otimes |b_i\rangle$$

if $i = j$.

This way, our symmetric basis has dimension $\dfrac{n(n+1)}{2}$ ($\dfrac{n(n-1)}{2}$ for $i \neq j$ and n for $i = j$).

Now we have the antisymmetric version where we only have $i < j$, dimensionality is $\dfrac{n(n-1)}{2}$. This also satisfies $\dfrac{n(n+1)}{2} + \dfrac{n(n-1)}{2} = n^2$; i.e., for the case of *two* identical particles (only), the subspace of symmetric wave functions plus the subspace of antisymmetric wave functions add up to the total 2-particle vector space.

We are also concerned with the spatial wave function. Back to the 2 spin 0 Boson scenario: The full two-particle vector space and its basis are

$$\mathbf{V}^{R3} \otimes \mathbf{V}^{R3};\ |\vec{r}\rangle \otimes |\vec{r}'\rangle$$

We begin again with the symmetric basis states $|\vec{r}, \vec{r}', S\rangle$. We avoid double counting by requiring $x' > x$ or ($x = x'$ and $y' > y$) or ($x = x', y = y'$ and $z' > z$). This yields two possibilities:

$$\vec{r} = \vec{r}' \rightarrow |\vec{r}, \vec{r}', S\rangle = |\vec{r}, \vec{r}\rangle$$
$$\vec{r} \neq \vec{r}' \rightarrow |\vec{r}, \vec{r}', S\rangle = \frac{1}{\sqrt{2}}(|\vec{r}, \vec{r}'\rangle + |\vec{r}', \vec{r}\rangle)$$

For some arbitrary, symmetric two-particle state vector $|\Psi_s\rangle$ we can define the "wave function" in configuration space in the usual way:

$$\Psi_s(\vec{r}, \vec{r}') = (\langle\vec{r}| \otimes \langle\vec{r}'|)|\Psi_s\rangle$$

where the ordering of the arguments indicates the "first" and "second" particle. Of course, since the state vector must be symmetric, we have $\Psi_s(\vec{r}, \vec{r}') = \Psi_s(\vec{r}', \vec{r})$.

This gets interesting when we write down the wave function in terms of the basis elements:

$$|\Psi_s\rangle = \int\int\int\int\int\int_{1/2space} d^3\vec{r} d^3\vec{r}' \langle\vec{r}, \vec{r}', S|\Psi_s\rangle |\vec{r}, \vec{r}', S\rangle$$

where the expression "1/2 space" reminds us that the integral is only to be taken over the range allowed by our convention for the basis vectors. (Alternatively one can integrate over all space and multiply with a factor 1/2). Here,

$$\langle\vec{r}, \vec{r}', S|\Psi_s\rangle = \frac{1}{\sqrt{2}}(\langle\vec{r}, \vec{r}'| + \langle\vec{r}', \vec{r}|)\Psi_s\rangle = \frac{1}{\sqrt{2}}(\Psi_s(\vec{r}, \vec{r}') + \Psi_s(\vec{r}', \vec{r})) = \sqrt{2}\Psi_s(\vec{r}, \vec{r}')$$

Normalization to 1:

$$1 = \langle\Psi_s|\Psi_s\rangle = \int\int\int\int\int\int_{1/2space} d^3\vec{r} d^3\vec{r}' |\langle\vec{r}, \vec{r}', S|\Psi_s\rangle|^2 =$$
$$\int\int\int\int\int\int_{1/2space} d^3\vec{r} d^3\vec{r}' 2|\Psi(\vec{r}, \vec{r}')|^2 = \int\int\int\int\int\int_{allspace} d^3\vec{r} d^3\vec{r}' |\Psi(\vec{r}, \vec{r}')|^2$$

which is the usual normalization condition for any vector in $\mathbf{V}^{R3} \otimes \mathbf{V}^{R3}$.

For antisymmetrtic spatial wave functions, everything works essentially the same; of course, we don't have to worry about the case $\vec{r} = \vec{r}'$, so the basis is simply given by

$$\vec{r} \neq \vec{r}' \rightarrow |\vec{r}, \vec{r}', A\rangle = \frac{1}{\sqrt{2}}(|\vec{r}, \vec{r}'\rangle - |\vec{r}', \vec{r}\rangle)$$

with the same conditions on the components of $\vec{r}, \vec{r}'$ as before. Of course, proper wave functions must now be antisymmetric, $\Psi_A(\vec{r}, \vec{r}') = -\Psi_A(\vec{r}', \vec{r})$. The normalization condition turns out to be identical.

4 Probabilities

Now assume I want to know the probability of finding particle 1 at $\vec{r}_1$ and particle 2 at $\vec{r}_2$ (out of a pair of identical particles). I have to find proper (anti-)symmetric vectors for the final state into which the wave function collapses, i.e. $\Psi_s(\vec{r}, \vec{r}')$ or $\Psi_A(\vec{r}, \vec{r}')$ (depending on whether the particles are Bosons or Fermions).

As an example, let's consider the symmetric case: The probability to find the two particles at the given positions, within volumes of size $d^3\vec{r}_1 d^3\vec{r}_2$, is

$$dProb(\vec{r}_1, \vec{r}_2, d^3\vec{r}_1 d^3\vec{r}_2) = |\langle \vec{r}_1, \vec{r}_2, S|\Psi_s\rangle|^2 d^3\vec{r}_1 d^3\vec{r}_2 = |\sqrt{2}\Psi(\vec{r}_1, \vec{r}_2)|^2 d^3\vec{r}_1 d^3\vec{r}_2$$

which is twice the probability of finding particle 1 at $\vec{r}_1$ and particle 2 at $\vec{r}_2$ (if they were distinguishable). Not surprising since there are two possible "classical" configurations that give the same measurement result: particle 1 at $\vec{r}_1$ and particle 2 at $\vec{r}_2$ or particle 2 at $\vec{r}_1$ and particle 1 at $\vec{r}_2$.