10/2/14
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$$i\hbar \frac{\partial}{\partial t}|\Psi\rangle = H|\Psi\rangle$$

$$\rightarrow i\hbar \frac{\partial}{\partial t}\Psi(x,t) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi(x,t) + V(x)\Psi(x,t)$$ Seeking special solution

remember light wave $\rightarrow e^{ikx} e^{-i\omega t}$ ? $\therefore$ Solution: $|\Psi\rangle = |\phi_E\rangle c(t)$ ← complex

$$i\hbar \frac{\partial}{\partial t}|\Psi\rangle = H|\Psi\rangle$$

$$i\hbar|\phi_E\rangle \frac{\partial}{\partial t}c(t) = \underbrace{H|\phi_E\rangle}_{E|\phi_E\rangle}c(t)$$

$$E|\phi_E\rangle ct$$

$$\Rightarrow i\hbar \frac{\partial}{\partial t}c(t) = E\,c(t) \longrightarrow \frac{\partial}{\partial t}c(t) = -i\frac{E}{\hbar}c(t)$$

$$c(t) = e^{-i\frac{E}{\hbar}t}$$

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi(x,t) + V(x)\Psi(x,t) \Rightarrow \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\phi_E(x) + V(x)\phi_E(x)$$
$$= E\phi_E(x)$$

$$\Rightarrow \boxed{\Psi(x,t) = \phi_E(x)e^{-i\frac{E}{\hbar}t}}$$ ← Stationary solution

Example 1.) $V(x) = 0$, $-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\phi_E(x) = E\phi_E(x)$

$$P : \phi_p(x) = e^{i\frac{p}{\hbar}x} \rightarrow p\,\phi_p(x)$$

$$P = \frac{\hbar}{i}\frac{\partial}{\partial x}$$

$$\frac{p^2}{2m}\phi_p(x) = \underbrace{\frac{p^2}{2m}}_{E}\phi_p(x)$$

$$\Psi_p(x,t) = A\cdot e^{i\frac{p}{\hbar}x} e^{-i\frac{E}{\hbar}t}$$ Solution for a free particle

$$H\cdot\Psi_p(x,t) = \frac{p^2}{2m}\Psi(x,t) = E\,\psi(x,t) = i\hbar\frac{\partial}{\partial t}\psi(x,t)$$ q.e.d.

expectation value (average) for measurement of $\Omega$

Operator $\Omega$, $\langle \Omega \rangle = \langle \Psi | \Omega | \Psi \rangle$, where $\Psi$ could be $\Psi(t)$

$$\langle \Psi(t) | \Omega | \Psi(t) \rangle = \int_{-\infty}^{\infty} \Psi^*(x,t)\, \Omega\, \Psi(x,t)\, dx$$

$$= \int_{-\infty}^{\infty} \phi_E^*(x)\, e^{i\frac{E}{\hbar}t}\, \Omega\, \phi_E(x)\, e^{-i\frac{E}{\hbar}t}\, dx$$

$$= e^{\left(i\frac{E}{\hbar}t - i\frac{E}{\hbar}t\right)} \int_{-\infty}^{\infty} \phi_E^*(x)\, \Omega\, \phi_E(x)\, dx$$

$$= \int_{-\infty}^{\infty} \phi_E^*(x)\, \Omega\, \phi_E(x)\, dx$$ no longer time-dependent $\Rightarrow$ stationary

~~GENERAL Solution for free case:~~

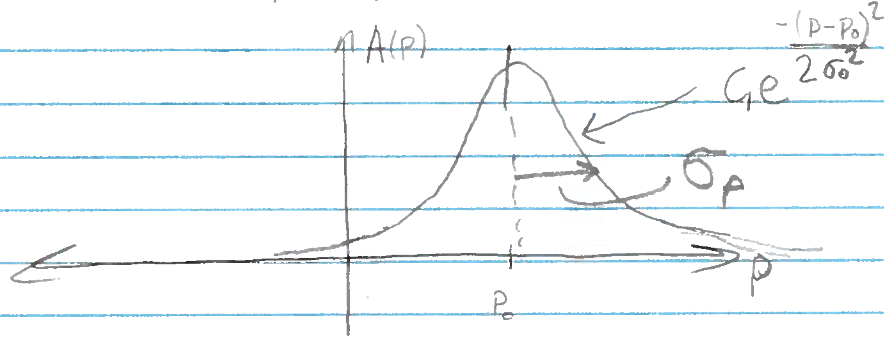
$$\Psi_{free}(x,t) = \int_{-\infty}^{\infty} dp\, A(p)\, \underbrace{e^{i\frac{px}{\hbar}}}_{\phi_E(x)}\, e^{-i\frac{p^2}{2m\hbar}t}$$ $\leftarrow$ mixture of different plane waves (Fourier decomposition)

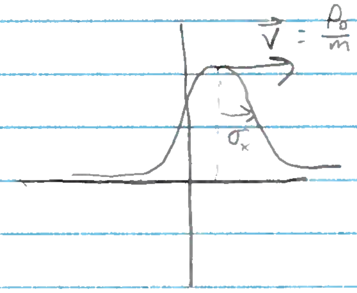
In general: $H\phi_E(x) = E\phi_E(x)$

$\Rightarrow \Psi(x,t) = \int \phi_E(x)\, e^{-iE/\hbar\, t}$

Not an eigenstate to $H$ (Hamiltonian operator)

Ex: Gaussian wave package

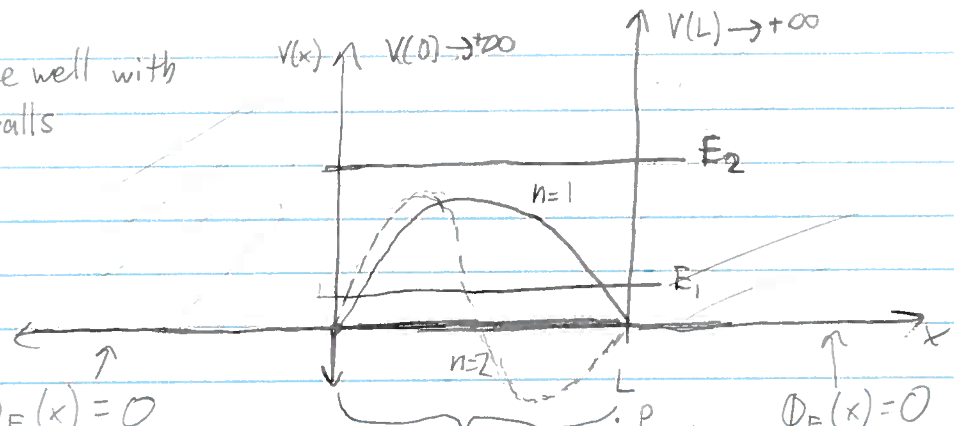



For the Gaussian wave package

$\sigma_x \cdot \sigma_p = \frac{\hbar}{2}$ for one point in time

1-D Square well with $\infty$ walls




$\phi_E(x) = 0$ $\quad -\infty < x < 0$

$$\phi_p(x) = Ae^{i\frac{p}{\hbar}x} + Be^{-i\frac{p}{\hbar}x}$$

$\phi_E(x) = 0$ $\quad 0 < x < +\infty$

$$E = \frac{p^2}{2m}$$

$\therefore A = -B$ so that $\phi_p(0) = 0$

$$\phi_E(x) = A\left(e^{i\frac{p}{\hbar}x} - e^{-i\frac{p}{\hbar}x}\right) \quad \text{[continuity!]}$$

$$\phi_E(x) = A\sin\left(\frac{p}{\hbar}x\right)$$

$$\sin \underbrace{\frac{p}{\hbar}L}_{n\pi} = 0 \quad \text{[continuity @ L!]}$$

choose $p_n = \dfrac{n\pi\hbar}{L}$

$$H\phi_{E_n}(x) = E_n \phi_{E_n}(x)$$

$$E_n = \frac{n^2\pi^2\hbar^2}{2mL^2} = \frac{p^2}{2m}$$

Note: there is a minimum $E_1 = \dfrac{\pi^2\hbar^2}{2mL^2}$, not 0!

Energy is quantized: $E_1$, $4E_1$, $9E_1$, ...