PHYS323 MODERN PHYSICS Lecture Notes 9/8/2015

**Summary up to now:**
Event (point) in space-time characterized by coordinates (ct, x, y, z) = ($x^\mu$) (contravariant 4-vector, $\mu$=0,1,2,3) with respect to some inertial coordinate system S.
Lorentz-transformation relates these to coordinates ($x'^\mu$) in some other inertial system S'.
Define ($x_\mu$) = (ct, -x, -y, -z) (covariant 4-vector) etc.

Separation between two space-time points (events): $\Delta x^\mu = x^\mu_2 - x^\mu_1$
Invariant squared interval: $\Delta s^2 = \Delta x^\mu \Delta x_\mu = \Delta ct^2 - \Delta x^2 - \Delta y^2 - \Delta z^2$ (repeated indices are summed over).
Important Law of Special Relativity: $\Delta s^2 = \Delta x^\mu \Delta x_\mu = \Delta s'^2 = \Delta x'^\mu \Delta x'_\mu$ !

Addition of velocities: S' moves with +v in +x-direction relative to S.
1) If object moves with velocity u' in +x'-direction relative to S', then its velocity (3-vector) in S is $\vec{u} = \left(\frac{u'+v}{1+u'v/c^2},0,0\right)$
2) If object moves with velocity u' in +y'=direction relative to S', then its velocity (3-vector) in S is $\vec{u} = \left(v,\frac{1}{\gamma}u',0\right)$; $\gamma = \frac{1}{\sqrt{1-v^2/c^2}}$

**NOW:** Continue on to DYNAMICS: Behavior of particles under the action of a force *F*!
Newton's Law: $\vec{F} = m\vec{a} = m\frac{d\vec{u}}{dt}$ if *m* is constant. More general, if *m* can vary (e.g., rocket burning its fuel or coal hopper being filled with coal), it is better to write $\vec{F} = \frac{dm\vec{u}}{dt} = \frac{d\vec{p}}{dt}$ with momentum **p** = *m***u**. We would like to figure out the correct definition of momentum in special relativity! At low velocities, it should be the same, but from experience we know that something might change at higher velocities. Assume that something is the mass (or, rather, the factor multiplying velocity). For now, let's use the "relativistic mass" *m*(*u*) which should only depend on the magnitude of **u** (to make all coordinate systems equivalent to each other). What can it be?

For this, we use **momentum conservation**: A system of masses on which there is no net **outside** force should have constant total momentum – in **all** Inertial Systems!

Consider the following example:
Our 2 usual coordinate systems S and S' represent a train (S') driving along a platform (S) in x-direction with high velocity *v*. Both on the train and on the platform there is one brother each of an identical pair of twins. Both brothers have a piece of putty in their hands which is exactly the same in size, mass, form etc. And they both hurl their projectiles at each other with exactly the same velocity (as measured relative to their OWN coordinate systems). Assume that relative to S, the putty of the 1st brother moves exactly in the y direction with velocity $-u_1$, and relative to S', the putty of the 2nd brother moves exactly in the y' direction with velocity $+u'_2$ ; both velocities have the same magnitude: $|u_1| = |u'_2|$. The two putty masses are $m_1$ (as measured in S) and $m'_2$ (as measured in S'), and again they must both be exactly equal in magnitude. In general, if the velocities $u_1$ and $u'_2$ are not small (relative to *c*), the two masses are not necessarily equal to the mass of putty at rest, $m_0$,, since we assumed

that the mass might be modified as a function of velocity. However, following the rule that relativity should “converge” to standard Newtonian mechanics when the velocities involved are small, we may assume that as $u_1 = u'_2 << c$, we have $m_1 = m'_2 = m_0$ .

Now let’s assume that the two putty masses collide in mid-flight, stick together and continue as a single mass. Because the situation is completely symmetric with respect to the y-axis, and since both coordinate systems must be equally “fundamental”, there is no way that the combined putty mass could have a velocity component in the y-direction - it can only move along x (or x’). Therefore, the final momentum component of the combined putty in y- (or y’-) direction is zero. Momentum conservation requires that therefore, the sum of the two y-components of the two initial putty masses must also be zero! This must be true for ANY coordinate system we choose to describe the whole process – momentum conservation must be coordinate-system independent!

Let’s describe the process from the point of view of S (you get the same result if instead you choose S’ – a nice little exercise!):
The y-component of the putty thrown by the 1st brother in S is of course $p_{y1} = -m_1 u_1$. Furthermore, from the summary above we know that the y-component of the velocity of the putty thrown by the 2nd brother, as measured in S, must be $u_2 = 1/\gamma\, u_2' = 1/\gamma\; u_1$. We must therefore conclude that the only way momentum conservation can work is if the mass $m_2$ of the 2nd putty, as measured in S, has increased by a factor $\gamma$ over that of the 1st one. Since this must be true no matter what the initial velocity $u'_2$ was, it follows that the velocity dependent mass of the putty must be equal to $m_2 = \gamma m_1 = \dfrac{m_2'}{\sqrt{1-v^2/c^2}}$.

This turns out to be universally true: The mass of any object increases by a factor $\Gamma$ from its rest mass $m_0$ once it is moving with velocity u: $m(u) = \dfrac{m_0}{\sqrt{1-u^2/c^2}} = \Gamma m_0 ; \Gamma = \dfrac{1}{\sqrt{1-u^2/c^2}}$ ,

and the momentum is therefore $\vec{p} = \dfrac{m_0}{\sqrt{1-u^2/c^2}}\vec{u} = \Gamma m_0 \vec{u}$ .

Let’s check if this works correctly even if the initial velocities $u_1 = u'_2$ are **not** small relative to $c$: In that case, the 2nd putty in S’ has already a mass larger than its rest mass:
$m_2' = \dfrac{m_0}{\sqrt{1-u_2'^2/c^2}}$. Of course, $m_1$ also must have this mass. Therefore, we must conclude that $m_2$ as measured in S has increased by another factor $\gamma$ (following the same argument above), and therefore It must be equal to

$$m_2 = \frac{m_2'}{\sqrt{1-v^2/c^2}} = \frac{m_0}{\sqrt{1-v^2/c^2}\sqrt{1-u_2'^2/c^2}} = \frac{m_0}{\sqrt{1-v^2/c^2-u_2'^2/c^2+v^2u_2'^2/c^4}} .$$

However, to be consistent, this same result should come out if we use the total velocity squared of the 2nd putty in S: $\vec{u}_2^2 = u_{2x}^2 + u_{2y}^2 = v^2 + \dfrac{1}{\gamma^2}u_2'^2 = v^2 + \left(1-v^2/c^2\right)u_2'^2$. It follows that the mass of the 2nd putty as measured in S **should** be equal to

$$m_2 = \frac{m_0}{\sqrt{1-u_2^2/c^2}} = \frac{m_0}{\sqrt{1-v^2/c^2+\left(1-v^2/c^2\right)u_2'^2/c^2}}$$

which is exactly the same result as we got before. So our conclusion is self-consistent. Furthermore, we can see that the transformation rules for transverse momenta are simple (as opposed to those for transverse velocities): $p_y = p_y'$! (The same must be true for the z-components).

Now let us turn to the x-components. Obviously, here things must be a bit more complicated. Let's study again a simple example involving the same train S' and platform S. But instead of two putties colliding, we now have a firecracker of initial mass $M'$ at rest in S' that explodes into two identical pieces, each with relativistic mass $m_1' = m_2'$. Let's assume that, just by chance, the two masses move with velocity $u'_1 = v$ and $u'_2 = -v$ along the x' axis, again all measured in S'. Clearly, momentum is conserved since $m_1' u'_1 + m_2' u'_2 = 0$, which is also the initial momentum along x (mass $M'$ at rest). Finally, we observe that if the mass of each fragment in its own rest frame is $m_0$, then $m_1' = \frac{m_0}{\sqrt{1-u_1'^2/c^2}} = \frac{m_0}{\sqrt{1-v^2/c^2}} = m_2'$

How does the whole explosion look like from the perspective of S? First, we can conclude that the mass $m_2$ of the fragment going backwards in S' must be equal to its rest mass $m_0$, since in fact $m_2$ is at rest in S ($u_2 = 0$), as one can see from the rule for velocity addition in x-direction. On the other hand, $m_1$ is moving with velocity $u_1 = \frac{u_1'+v}{1+u_1'v/c^2} = \frac{2v}{1+v^2/c^2}$, so its mass must be equal to

$$m_1 = \frac{m_0}{\sqrt{1-u_1^2/c^2}} = \frac{m_0}{\sqrt{1-\left(\frac{2v}{1+v^2/c^2}\right)^2/c^2}} = \frac{m_0\left(1+v^2/c^2\right)}{\sqrt{\left(1+v^2/c^2\right)^2-(2v)^2/c^2}} = m_0\frac{1+v^2/c^2}{1-v^2/c^2}.$$

Finally, we can conclude (again from our knowledge of how masses depend on velocity) that the original mass of the firecracker, as measured in S (where it has velocity $v$), must be $M = \frac{M'}{\sqrt{1-v^2/c^2}}$. Now we can impose momentum conservation in x-direction in S:

$$p_1 + p_2 = m_1 u_1 + 0 = m_0\frac{1+v^2/c^2}{1-v^2/c^2}\frac{2v}{1+v^2/c^2} = \frac{2m_0 v}{1-v^2/c^2} \overset{!}{=} Mv = \frac{M'v}{\sqrt{1-v^2/c^2}}.$$

By direct comparison, we must conclude that $M = \frac{2m_0}{1-v^2/c^2} = m_0\frac{1+v^2/c^2}{1-v^2/c^2} + m_0\frac{1-v^2/c^2}{1-v^2/c^2} = m_1 + m_2$ !

Amazingly, it appears that whatever the relativistic replacement of "mass" is, is actually conserved in a collision, as well – NOT only the three components of the moment (and NOT the rest masses as would be the case in Newtonian mechanics: $M \neq 2m_0$ in either coordinate system!) Again, as a consistency check, we can now deduce that

$$M' = \sqrt{1-v^2/c^2}\,M = \frac{2m_0}{1-v^2/c^2}\sqrt{1-v^2/c^2} = 2\frac{m_0}{\sqrt{1-v^2/c^2}} = m_1' + m_2',$$

so the same relativistic mass conservation is true in S', as well. Obviously, there must be some deeper meaning to the relativistic mass $m(u)$ that we introduced!

To figure out what this meaning is, let's see what the limit of the relativistic mass is at small but non-zero velocities, where Newton mechanics should apply approximately. Therefore, we develop the expression for the relativistic mass into a Taylor series:

$$m(u) = \frac{m_0}{\sqrt{1-u^2/c^2}} \approx m_0\left(1-\frac{1}{2}\left[-u^2/c^2\right]+\frac{3}{8}\left[-u^2/c^2\right]^2+\cdots\right) \approx \frac{1}{c^2}\left(m_0 + \frac{m_0}{2}u^2\right)+\cdots$$

The first term is indeed the normal (proper) rest mass, but the second term turns out to be the kinetic energy (divided by $c^2$). So, we find that, to our surprise, the conserved quantity is something like a combination of mass and energy! Since energy is another quantity that **should** be conserved in any collision, Einstein concluded that the mass $m(u)$, multiplied with $c^2$, must be equal to a more general definition of energy:

$E_{tot,rel} = m(u)c^2 = \dfrac{m_0 c^2}{\sqrt{1-u^2/c^2}} = \Gamma m_0 c^2 = m_0 c^2 + T_{kin}$. NOTE: **Only** for small velocities can $T_{\text{kin}}$ be approximated by $m_0 u^2/2$– at higher velocities, it is **defined** as the difference $E_{\text{tot}} - m_0 c^2 = (\Gamma\text{-}1)\, m_0 c^2$. To summarize: The sum of all actual (rest) masses of objects in a collision is no longer conserved in Special Relativity. However, if we generalize energy by adding the so-called rest-mass energy $m_0\, c^2$ (and using the relativistically correct form of kinetic energy), we end up once again with a conserved quantity – the total relativistic energy $E_{\text{tot}} = \Gamma m_0 c^2$.

Obviously, there are very important consequences to this identification which we will explore in detail. However, before we get to this, let's conclude our analysis of how momenta transform from one coordinate system to another. Let's again use our coordinate systems S and S', but now take an arbitrary particle with rest mass m0 moving with velocity u' in the x-direction relative to S'. What is its momentum in S'? $p_x' = m'(u')u' = \dfrac{m_0 u'}{\sqrt{1-u'^2/c^2}}$.

Of course, it's speed in S is $u = \dfrac{u'+v}{1+u'v/c^2}$ and therefore its mass is

$$m(u) = \frac{m_0}{\sqrt{1-u^2/c^2}} = \frac{m_0}{\sqrt{1-\left(\dfrac{u'+v}{1+u'v/c^2}\right)^2/c^2}} = \frac{m_0\left(1+u'v/c^2\right)}{\sqrt{\left(1+u'v/c^2\right)^2-\left(u'+v\right)^2/c^2}} =$$

$$= \frac{m_0\left(1+u'v/c^2\right)}{\sqrt{1+u'^2v^2/c^4-u'^2/c^2-v^2/c^2}} = \frac{m_0\left(1+u'v/c^2\right)}{\sqrt{\left(1-u'^2/c^2\right)\left(1-v^2/c^2\right)}}\left[=\gamma m' + \gamma\frac{v}{c}\frac{p_x'}{c}\right]$$

and its momentum in x-direction

$$p_x = m(u)u = \frac{m_0\left(1+u'v/c^2\right)}{\sqrt{\left(1-u'^2/c^2\right)\left(1-v^2/c^2\right)}}\frac{u'+v}{1+u'v/c^2} = \frac{m_0\left(u'+v\right)}{\sqrt{\left(1-v^2/c^2\right)}\sqrt{\left(1-u'^2/c^2\right)}}$$

$$= \gamma p_x' + \gamma v m' = \gamma p_x' + \gamma\frac{v}{c}\frac{E'}{c}$$

where we have used Einstein's equation in S' : $E' = m'(u')\, c^2$. This looks suspiciously similar to the Lorentz transformations for $x^\mu$ at the beginning of this LN! In fact, if we define a "zero-component of the momentum vector" as $p^0 = E/c = m(u)c$, then we can write

$p^0 = m(u)c = \gamma m'c + \gamma\frac{v}{c}p_x' = \gamma p'^0 + \gamma\frac{v}{c}p'^1$ (from the [] brackets 2 equations above) and

$p^1 = p_x = \gamma p_x' + \gamma\frac{v}{c}\frac{E'}{c} = \gamma p'^1 + \gamma\frac{v}{c}p'^0$. We even have a relativistic invariant:

$p^\mu p_\mu = \left(p^0\right)^2 - \left(p_x\right)^2 - \left(p_y\right)^2 - \left(p_z\right)^2 = \gamma^2 m_0^2 c^2 - \gamma^2 m_0^2 v^2 = m_0^2 c^2$ (true in all coordinate systems)