Introduction: Special Relativity

- Observation: The speed *c* (e.g., the speed of light) is the same in all coordinate systems (i.e. an object moving with *c* in **S** will be moving with *c* in **S**')
- Therefore: If $\left|\Delta\vec{r}\right| = c\Delta t \Rightarrow \quad (c\Delta t)^2 - (\Delta\vec{r})^2 = 0$ is valid in one coordinate system, it should be valid in all coordinate systems!
- => Introduce 4-dimensional "space-time" coordinates:
  $x^0 = ct; \left(x^1, x^2, x^3\right) = \vec{r}$
- => Introduce "metric" g that defines the "distance" between any 2 space-time points (using Einstein's summation convention) as
  $(\Delta s)^2 = g_{\mu\nu}\Delta x^\mu \Delta x^\nu; \quad g_{00} = 1, g_{11} = g_{22} = g_{33} = -1, \text{all others } = 0$
- Postulate that all products between 2 vectors and the metric is invariant (the same in all coordinate systems)
- Meaning? If |Δct| > Δ**r**, for a moving object, then there is one system S_o where Δ**r** = 0 => rest frame for that object. => $\sqrt{(ds)^2}$ is the time elapsed in S_0 between the 2 points ("Eigentime")

Examples

- Object moving (relative to **S**) with speed *v* along *x*. "Distance" between point 1 = (0,0,0,0) (origin) and point 2 = (*ct*, *vt*, 0, 0):
  $$(\Delta s)^2 = g_{00}(ct)^2 + g_{11}(vt)^2 + g_{22}0^2 + g_{33}0^2 = (ct)^2 - (vt)^2 = (c\tau)^2$$
  where τ is the "eigentime" (time elapsed between the two points in the frame S_0 where the object is at rest - i.e. the system moving with v along x-axis) => $\tau = t \cdot \sqrt{1 - v^2/c^2} = \gamma^{-1} t$
- Consequence: As seen from S, the clock in S_o is "going slow"!
  - From point of view of S_o, it is the clock in S that is going slow!
  - With similar arguments, one can prove "length contraction", "relativity of synchronicity" and all the other "relativity weirdness"
- Argument can be extended to other quantities: All must come as 4-vectors or as invariant scalars (or tensors…), and the same metric applies to calculate the "invariant length" of each 4-vector
  - Example: 4-momentum $p^\mu = \left(\frac{E}{c}, \vec{p}\right); g_{\mu\nu} p^\mu p^\nu = \left(\frac{E}{c}\right)^2 - \vec{p}^2 = m^2 c^2$

Now a bit more General…

- Equivalence Principle: Motion in a gravitational field is (locally) indistinguishable to force-free motion in accelerated coordinate system S': $y = -\,{}^1/_2 g t^2$
  - Example: Free fall in elevator
  - 2nd example: Clock moving around circle with radius r, angular velocity ω, speed $v=r\omega$ => goes slow by factor $\sqrt{1-r^2\omega^2/c^2} = \sqrt{1-r"g"/c^2}$ where "g"$=r\omega^2$ is the centripetal acceleration. If we replace this with a gravitational force, we must choose $\Phi = U_{pot}/m$ such that $d\Phi/dr = -r\omega^2$ => $\Phi = -{}^1/_2 r^2\omega^2$

$$\Rightarrow \quad \tau_{clock} = \sqrt{1+2\Phi(r)/c^2}\, t$$

- => New metric: $g_{00} = 1+2\Phi/c^2$
  - Example: clock at bottom of 50 m tower is slower by $5\cdot 10^{-15}$ than clock at top. Can be measured using Mößbauer effect!
  - More general: Curved space-time!

1 foot!

General Relativity

- Einstein's idea: Space-time is curved, with a metric determined by mass-energy density
- Force-free objects move along geodetics: paths that maximize elapsed eigentime (as measured by metric)
  - Example: twin paradox – it is really the stay-at-home twin that ages more
- Most general equations complicated (differential geometry), but special manageable case: spherically symmetric mass *M* at rest => Schwarzschild metric

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

$$(ds)^2 = \left(1 - \frac{2GM}{rc^2}\right)(cdt)^2 - \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2 - r^2\left[(d\theta)^2 + \sin^2\theta(d\phi)^2\right]$$

- Examples: Falling, redshifting, bending, gravitational lensing, <u>Event horizon (Schwarzschild radius)</u>

Calculation: Falling

Radial motion Δr in 2 steps (each taking time Δt): Δr_1, $\Delta r_2 = \Delta r - \Delta r_1$

$$\Delta s \approx \sqrt{\left(1+\frac{2\Phi_1}{c^2}\right)(\Delta ct)^2 - (\Delta r_1)^2} + \sqrt{\left(1+\frac{2\Phi_2}{c^2}\right)(\Delta ct)^2 - (\Delta r_2)^2} = \Delta ct\left[\sqrt{1+\frac{2\Phi_1}{c^2}-\left(\frac{\Delta r_1}{\Delta ct}\right)^2} + \sqrt{1+\frac{2\Phi_2}{c^2}-\left(\frac{\Delta r_2}{\Delta ct}\right)^2}\right]$$

$$\approx \Delta ct\left[1+\frac{\Phi_1}{c^2}-\frac{1}{2}\left(\frac{\Delta r_1}{\Delta ct}\right)^2 + 1 + \frac{\Phi_2}{c^2}-\frac{1}{2}\left(\frac{\Delta r_2}{\Delta ct}\right)^2\right] \approx \Delta ct\left[2+\frac{\Phi_0+\frac{d\Phi}{dr}\frac{\Delta r_1}{2}}{c^2}-\frac{1}{2}\left(\frac{\Delta r_1}{\Delta ct}\right)^2 + \frac{\Phi_0+\frac{d\Phi}{dr}\frac{\Delta r+\Delta r_1}{2}}{c^2}-\frac{1}{2}\left(\frac{\Delta r_2}{\Delta ct}\right)^2\right]$$

Find extremum of Δs w.r.t. Δr_1 (for which Δr_1 does Δs become max.?):

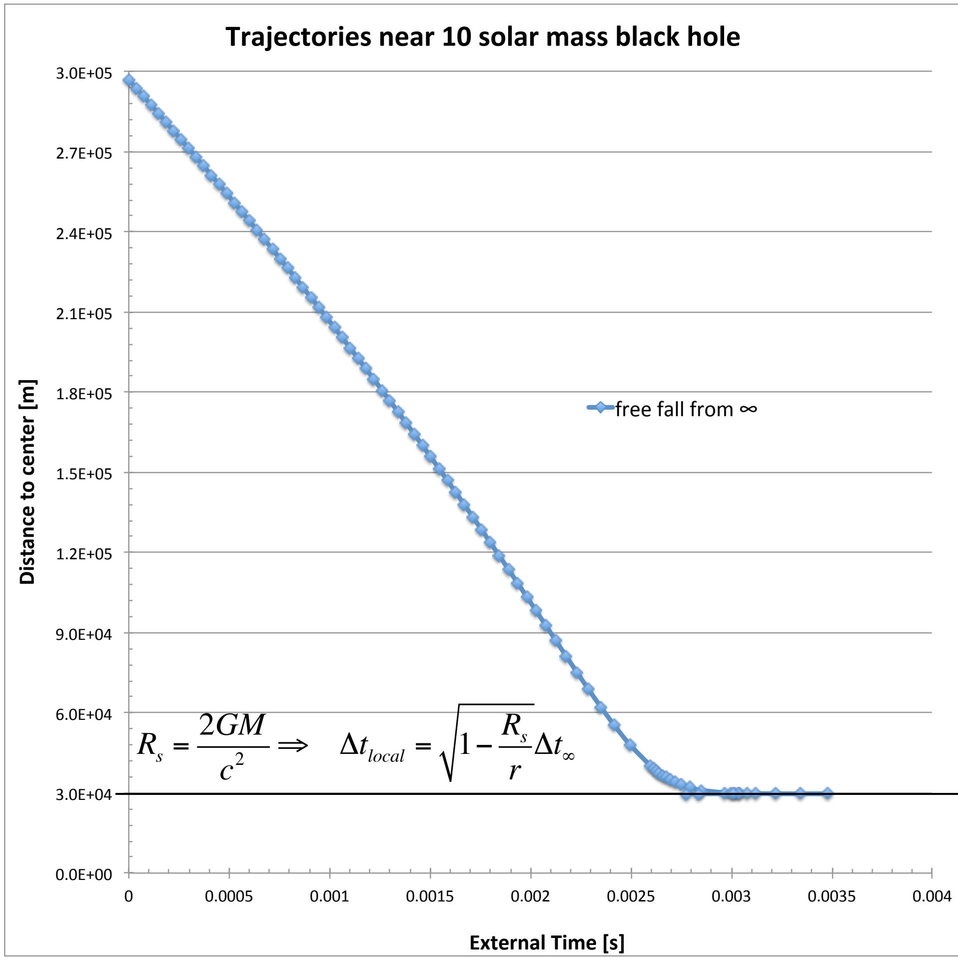
$$\frac{d\Delta s}{d\Delta r_1} \overset{!}{=} 0 \Rightarrow \quad 0 = \frac{\frac{1}{2}\frac{d\Phi}{dr}}{c^2} - \frac{\Delta r_1}{(\Delta ct)^2} + \frac{\frac{1}{2}\frac{d\Phi}{dr}}{c^2} - \frac{\Delta r_2(-1)}{(\Delta ct)^2} = \frac{1}{\Delta ct}\left[\frac{\Delta r_2}{\Delta ct}-\frac{\Delta r_1}{\Delta ct}\right] + \frac{1}{c^2}\frac{d\Phi}{dr}$$

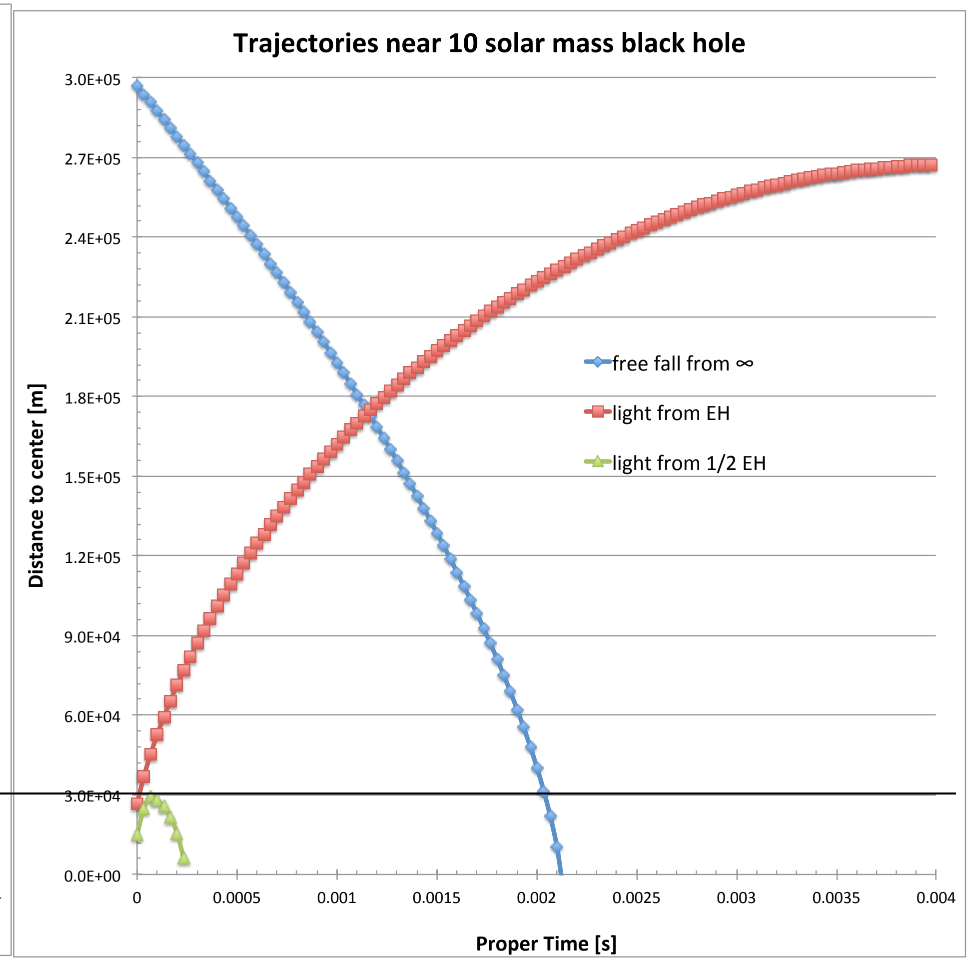
$$= \frac{1}{c^2}\left[\frac{1}{\Delta t}(v_2 - v_1) + \frac{d\Phi}{dr}\right] \Rightarrow \quad a = -\frac{d\Phi}{dr} \text{ q.e.d.}$$

=> Black Holes

- Beyond a certain density, NOTHING can prevent gravitational collapse!
  - If there were a new source of pressure, that pressure would have energy (P = 1/3-2/3 u), which causes more gravitation => gravity wins over
  - Singularity in space-time (infinitely dense mass point, infinite curvature; no classical treatment possible)
- For spherical mass at rest, Schwarzschild metric applies and we have an event horizon at $r = R_S = 2GM/c^2$ = 3km M/M_{sun} (Schwarzschild radius)
  - as object approaches r_S from outside, clock appears to slow to a crawl and light emitted gets redshifted to ∞ long wavelength
  - along light path, $ds = 0$ => $dr = \pm(1-r_S/r)\cdot dct$ => light becomes ∞ slow and never can cross from inside r_S to outside
  - From outside, it takes exponential time for star surface to reach r_S
  - Rate of photon emission decreases exponentially (less than 1/s after 10 ms)
  - All material that falls in over time "appears" frozen on the surface of event horizon but doesn't emit any photons or any other information
  - Co-moving coordinate system: will cross event horizon in finite time => no return!

Sample trajectories



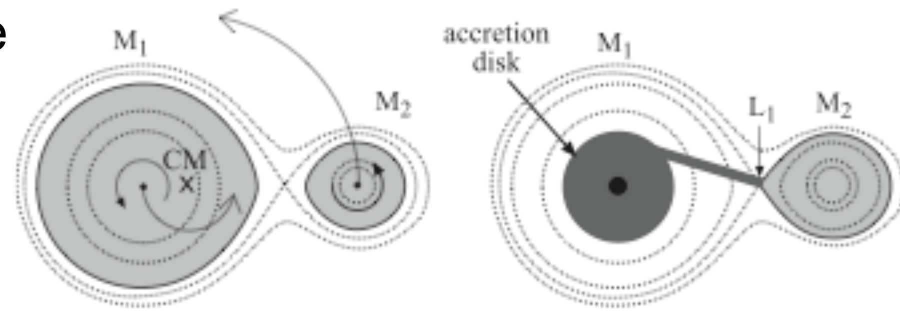


$$\Delta t_{local} = \sqrt{1 + \frac{2\Phi_{grav}}{c^2}} \Delta t_\infty \,; \Phi_{grav} = \frac{V_{pot}^{grav}}{m} \left(= -\frac{GM}{r} \right)$$

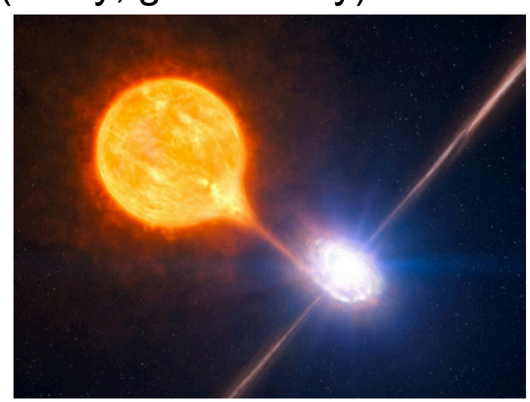
$$d\tau = \left(\left[1 - \frac{R_s}{r} \right] dt_\infty^2 - \left[1 - \frac{R_s}{r} \right]^{-1} \frac{dr^2}{c^2} \right)^{1/2}$$

Black holes in the Wild

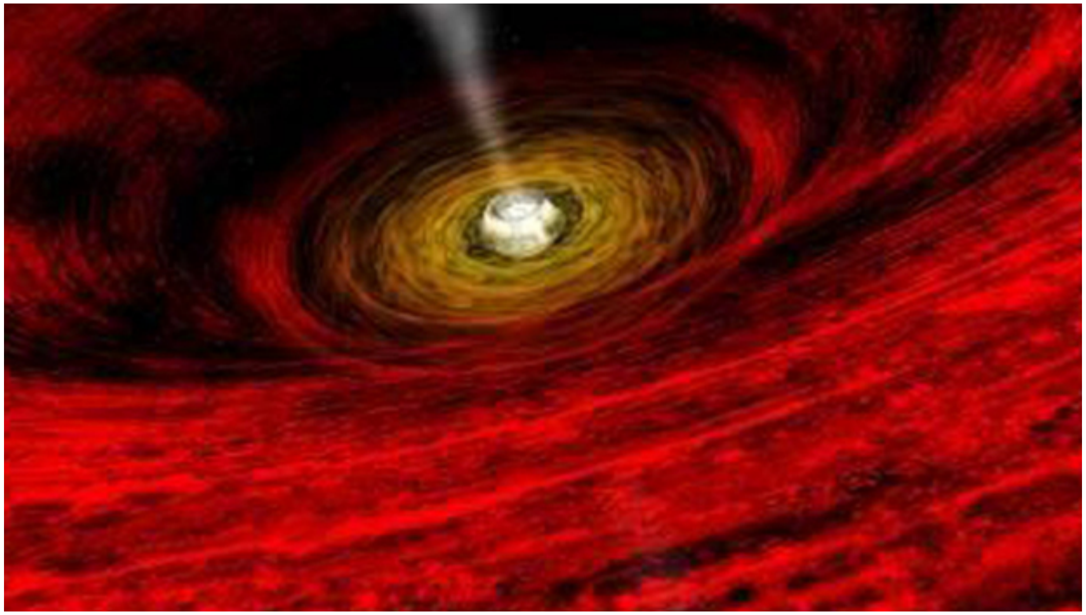
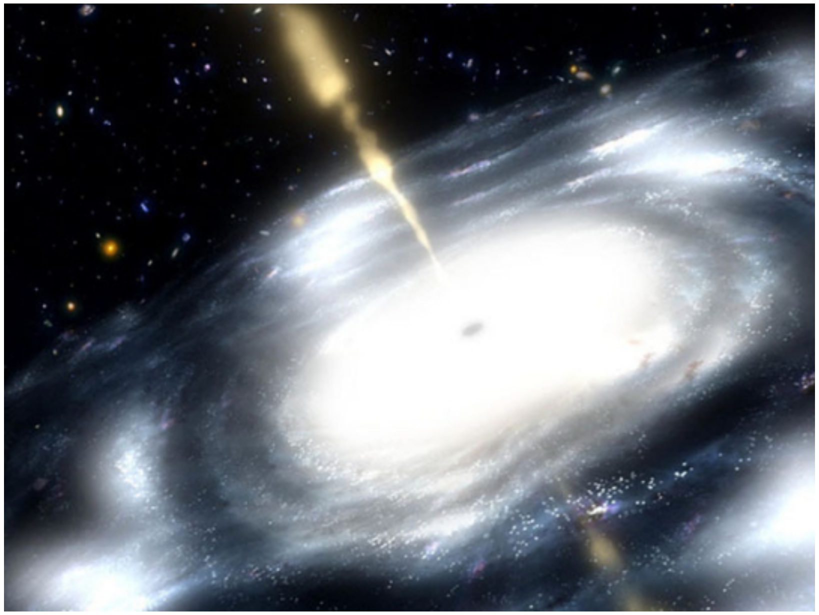
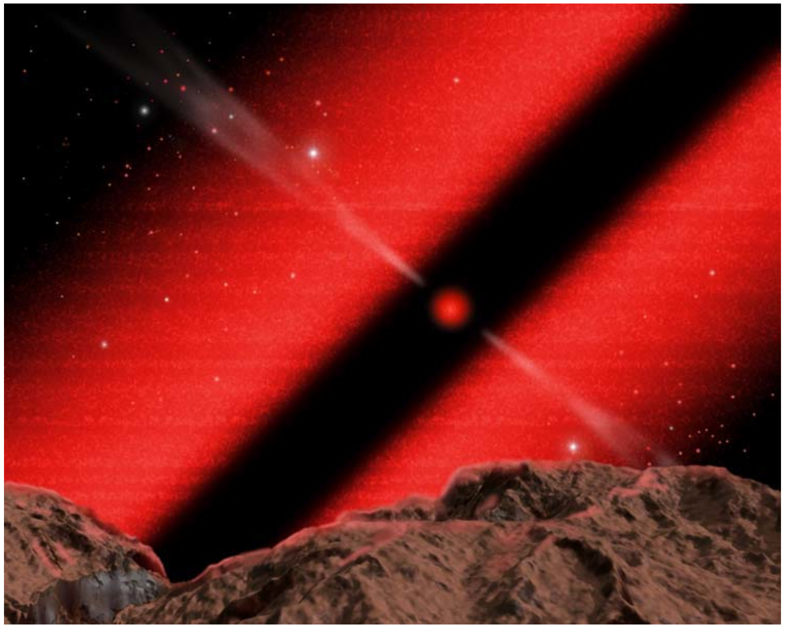
- Smallest black holes likely > 3 M_{sun} (supernovae of 25 M_{sun} star followed by complete core collapse)
  - mostly detectable as invisible partner in binary system
  - some radiation from accretion disks (esp. X-ray) smallest radius about 3 r_S; about 5-10% of gravitational pot. energy gets converted into luminosity (much more than fusion in stars in case of ns/bh)
    - White dwarf: $L \cong L_{sun}$ (UV); ns/bh 1000's times more (X-ray, gamma-ray)



- Gigantonormous black holes in center of galaxies (see later in semester)
- Primordial black holes?



Black holes in the Wild


Gravitational Waves