Section 2 – 1st Homework Problem Set - Solution

Problem 1

Depending on how you account for the additional energy due to positron annihilation and the energy loss from escaping neutrinos, the pp chain produces about 26 MeV for one Helium nucleus formed; that is 6.5 MeV per initial proton. On the other hand, the "triple-alpha" reaction releases 7.28 MeV of energy – that's a tad over 0.6 MeV per each of the 12 nucleons involved. From this result alone I would expect a 10-11 times shorter "burn time" at equal luminosity for ^{4}He burning vs. hydrogen burning.

Problem 2

Let's assume the surface of area *A* is oriented perpendicular to the x-axis. Then the number of particles with velocity component $v_x > 0$ that "crash" into *A* during the time d*t* is $dN = nAv_x dt$ (i.e., all particles within the volume spanned by the surface area A and the "depth" $v_x dt$ = the distance from which they can reach *A* during d*t*). Since they "flip" the x-component of their momentum, p_x, upon impact, the momentum transferred by each particle is 2 p_x. Therefore, the total momentum transfer onto the area *A* during the time d*t* is $dp_x = 2p_x dN = 2p_x nAv_x dt$. Dividing by d*t* gives the momentum transfer per unit time, i.e. the force on *A*, and dividing by *A* gives the pressure: $P = \frac{1}{A}\frac{dp_x}{dt} = 2np_x v_x$. Now we have to average this over all particles (i.e. over all values of v_x and p_x). We observe that $\vec{v} \cdot \vec{p} = v_x p_x + v_y p_y + v_z p_z$ and if we assume that the gas is isotropic, then each of the 3 terms will contribute 1/3 of the left hand side. However, we need to divide by another factor 2 because only that half of the contribution to $v_x \cdot p_x$ for which both v_x and p_x are positive contributes to the pressure on *A* (from one side). So we get the final result $P = n\frac{1}{3}\langle \vec{v} \cdot \vec{p} \rangle$ where the angle brackets indicate that one is to take the average over all particles.

Now we have to distinguish between non-relativistic and ultra-relativistic particles. In the first case, we notice that $\vec{v} \cdot \vec{p} = 2E_{kin}$. In the second case, the magnitude of v is simply equal to c. From the lecture, we recall that the resulting expression, pc, is equal to the kinetic energy in the ultra-relativistic case. So we get for our 2 cases:

$$P = \begin{cases} n\frac{2}{3}\langle E_{kin}\rangle \text{; non-relativistic} \\ n\frac{1}{3}\langle E_{kin}\rangle \text{; ultra-relativistic} \end{cases}$$

Now all we have to do is plug in the results from the lecture for the average energy of an electron in a Fermi gas:

$$\langle E_{kin}\rangle = \frac{E_{tot}}{N_{tot}} = \begin{cases} \frac{3}{10m}\left(3\pi^2\right)^{2/3} n^{2/3}\hbar^2 \text{ ;non-rel.} \\ \frac{3}{4}\left(3\pi^2\right)^{1/3} n^{1/3}\hbar c \text{ ; ultra-rel.} \end{cases}$$

and we get

$$P = \begin{cases} n\frac{2}{3}\frac{3}{10m}\left(3\pi^2\right)^{2/3} n^{2/3}\hbar^2 = \frac{1}{5m}\left(3\pi^2\right)^{2/3} n^{5/3}\hbar^2 \text{; non-rel.} \\ n\frac{1}{3}\frac{3}{4}\left(3\pi^2\right)^{1/3} n^{1/3}\hbar c = \frac{1}{4}\left(3\pi^2\right)^{1/3} n^{4/3}\hbar c \text{; ultra-rel.} \end{cases}$$

which is the result in the lecture notes, q.e.d.

Problem 3

Taking our result from Problem 1, $P = \frac{1}{4}\left(3\pi^2\right)^{1/3} n^{4/3}\hbar c$, and n=10^{36}/m^3 and $\hbar c = 1.055 \cdot 10^{-34}$ Js $\times$ $2.9979 \cdot 10^8$ m/s = $3.16 \cdot 10^{-26}$ Jm, I get P = $2.445 \cdot 10^{22}$N/m^2.

On the other hand, the central pressure from the lecture is $P(0) \approx \frac{2\pi G\rho^2 R^2}{3}$.

Plugging in all the constants ($\rho = 3 \cdot 10^9$ kg/m^3, R = 5,564 km) yields $3.89 \cdot 10^{22}$N/m^2. Thus it seems like the Fermi-gas pressure is a little too low to keep Sirius B from collapsing further, but this is only because of the many oversimplifications I've made.

If the density increases by a factor 8, then the Fermi gas pressure will increase by a factor $8^{4/3}$ = 16, so it would seem that now the Fermi gas pressure should be enough. However, the density ρ will also go up by a factor 8, while the radius will go down by a factor 2, so the central pressure P(0) will also go up by a factor 16. This means that if the relativistic Fermi gas pressure is not enough to prevent collapse at Sirius B density, it also won't be enough at 8 times the density (1/2 the radius), or, for that matter, at **any** density! So there is nothing to stop collapse to arbitrarily small radius (except of course for the Fermi gas pressure of **neutrons**).