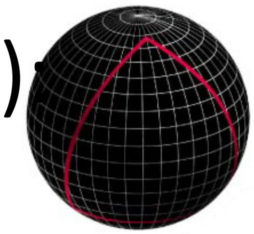
Walker-Robertson metric

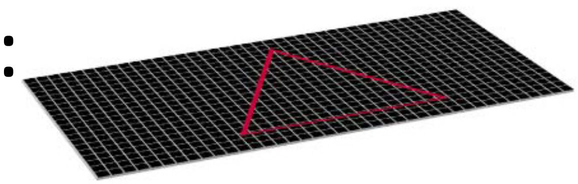
$$ds^2 = dt^2 - a^2(t)\left[dr_c^2 + S_K^2(r_c)\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)\right]$$

- Closed Universe, positive curvature (K = +1):



$$S_K(r_c) = \sin(r_c)$$

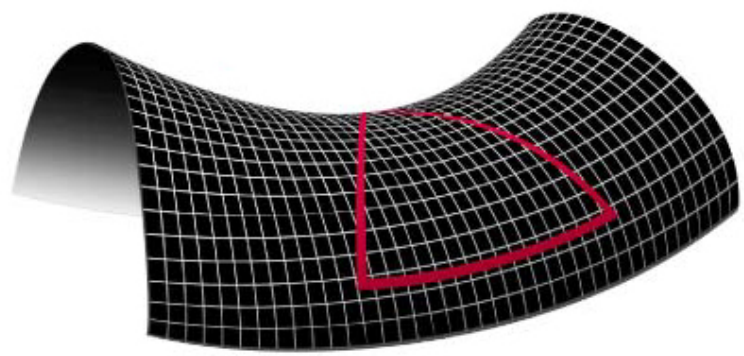
- Flat Universe, no curvature (K = 0):



$$S_K(r_c) = r_c$$

- Open Universe, negative curvature (K = -1):

$$S_K(r_c) = \sinh(r_c)$$



Important equations

Hubble law: $v_r = \frac{dD}{dt} = \dot{a}(t) r_c = \frac{\dot{a}(t)}{a(t)} D(t) =: H(t) D$. At present:

$H_0 = H(t_0) = \frac{68-70 \text{ km/s}}{1 \text{ Mpc}} \approx \frac{1}{14 \cdot 10^9 \text{ yr}}$. Speed of light in co-moving coordinates: $\frac{dr_c}{dt} = \frac{c}{a(t)}$

Redshift for light emitted at t and received at t_0: $z = \frac{a(t_0)}{a(t)} - 1$. Invariant distance of object at time of emission: $r_c(em.) = \int_t^{t_0} \frac{c}{a(t')} dt' \Rightarrow D(em.,t) = a(t) r_c(em.); D(em.,\text{today}) = a(t_0) r_c(em.)$.

General (Walker-Robertson) metric: $ds^2 = dt^2 - a^2(t)\left[dr_c^2 + S_K^2(r_c)\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)\right]$.

Transformation to any other local coordinate system according to Special Relativity.

Closed Universe, positive curvature (K = +1):

$S_K(r_c) = \sin(r_c)$

Flat Universe, no curvature (K = 0):

$S_K(r_c) = r_c$

Open Universe, negative curvature (K = -1):

$S_K(r_c) = \sinh(r_c)$

$a(t)$ = |Radius of curvature|

Important equations – questions:

Note: in the following slides, $t = t_0 = 0$ refers to ”today”

1. What determines $a(t)$?
2. What determines $K = -1, 0$ or 1?
3. What are the initial conditions?

Escape velocity of mass m at some distance $r = a(t) \cdot r_c$ from “center”:

$$\frac{m}{2}v^2 = \frac{GMm}{r} = \frac{G4\pi r^3 \rho m}{3r} \Rightarrow v^2 = \frac{8\pi r^2 G\rho}{3} \Rightarrow \dot{a}^2 = a^2\frac{8\pi G\rho}{3} \Rightarrow \rho = \frac{3}{8\pi G}H^2$$

since $r = a\, r_c$ and $v = da/dt\, r_c$

Critical density: $\rho_c(t) = \dfrac{3H^2(t)}{8\pi G}$. Calculate today’s value: 10^{-26} kg/m^3 = 6 protons

More general: $$E = \frac{m}{2}v^2 - \frac{GMm}{r} \Rightarrow \frac{E}{T_{kin}} = 1 - \frac{8\pi G\rho r^2}{3v^2} = 1 - \frac{8\pi G\rho}{3H^2(t)} = 1 - \frac{\rho(t)}{\rho_c(t)}$$

Einstein: $H^2\dfrac{E}{T_{kin}} = -\dfrac{kc^2}{a^2(t)};\ k = -1,0,1$ k = sign and $a(t)$ = radius of curvature

$$\Rightarrow H^2(t) = H^2(t)\frac{\rho(t)}{\rho_c(t)} - \frac{kc^2}{a^2(t)} = H^2(0)\left(\frac{\rho(t)}{\rho_c(0)} - \frac{kc^2}{H^2(0)a^2(t)}\right)$$

Consequences:

Found: Critical density: $\rho_c(t) = \dfrac{3H^2(t)}{8\pi G}$. $H^2(t) = H_0^2\left(\dfrac{\rho(t)}{\rho_c(0)} - \dfrac{kc^2}{H_0^2 a^2(t)}\right)$

Closed Universe (positive curvature): $\rho_{tot} = \rho_M + \rho_R + \rho_\Lambda > \rho_c \Rightarrow K = 1, S_K(r_c) = \sin(r_c)$.

Flat Universe (no curvature): $\rho_{tot} = \rho_c \Rightarrow K = 0, S_K(r_c) = r_c$.

Open Universe (negative curvature): $\rho_{tot} < \rho_c \Rightarrow K = -1, S_K(r_c) = \sinh(r_c)$.

Note: k, c, H_0 are constants and $a^2(t) > 0 \Rightarrow$ once open(closed, flat), always open (c,f)

Contributions to density (r.h.s.):

1. Matter (cold, massive particles
2. Radiation (ultrarelativistic particles)
3. Dark energy (cosmological constant
4. Curvature (intrinsic property of universe)

$$\Omega_M(t) = \frac{\rho_M(t)}{\rho_c(0)}$$

$$\Omega_R(t) = \frac{\rho_R(t)}{\rho_c(0)} = \frac{<\varepsilon_R>}{\rho_c(0)c^2}$$

$$\Omega_\Lambda(t) = \frac{\rho_\Lambda(t)}{\rho_c(0)} = \frac{<\varepsilon_\Lambda>}{\rho_c(0)c^2}$$

$$\Omega_K(t) = -\frac{kc^2}{H_0^2 a^2(t)}$$

Sum of all 4 must be equal to 1 at $t = 0$

Important equations 2

Evolution Equation:

$$H^2(t) = \frac{\dot{a}^2(t)}{a^2(t)} = \frac{8\pi}{3} G\rho_{tot}(t) - \frac{Kc^2}{a^2(t)} = H_0^2\left(\frac{\rho_{tot}(t)}{\rho_c(t_0)} - \frac{Kc^2}{H_0^2 a^2(t)}\right) = H_0^2\left(\Omega_M(t) + \Omega_R(t) + \Omega_\Lambda(t) + \Omega_K(t)\right)$$

$$\Rightarrow \dot{a}(t) = a(t) H_0 \sqrt{\left(\Omega_M(t) + \Omega_R(t) + \Omega_\Lambda(t) + \Omega_K(t)\right)} \Rightarrow \frac{da}{a_0} = \sqrt{\frac{a^2}{a_0^2}\left(\Omega_M(t) + \Omega_R(t) + \Omega_\Lambda(t) + \Omega_K(t)\right)}\, H_0 dt$$

with the following ingredients:

1) Matter (both baryonic and dark matter), non-relativistic, density due to mass: $\Omega_M(t) = \frac{\rho_M(t)}{\rho_c(t_0)} = \frac{\rho_M(t_0)}{\rho_c(t_0)} \frac{a_0^3}{a^3(t)} = \Omega_M^0 \frac{a_0^3}{a^3}$. (Note the "mixed defintion" where ρ_c is always taken at today's value). Today: $\Omega_M^0 \approx 0.3$, roughly 26% dark matter and 4% baryons.

2) Radiation (any relativistic particles, including photons, neutrinos in the early Universe and ultra-hot matter): $\Omega_R(t) = \frac{\rho_R(t)}{\rho_c(t_0)} = \frac{\varepsilon(t)/c^2}{\rho_c(t_0)} = \frac{\rho_R(t_0)}{\rho_c(t_0)} \frac{a_0^4}{a^4(t)} = \Omega_R^0 \frac{a_0^4}{a^4}$.

$\varepsilon(t)$ = energy density $\propto T^4; T(t) = T\left(t_0\right)\frac{a_0}{a(t)}$. Today: $\Omega_R^0 = 8.24 \cdot 10^{-5}$ (mostly due to CMB)

3) Dark energy (cosmological constant Λ): $\Omega_\Lambda(t) = \frac{\rho_\Lambda(t)}{\rho_c(t_0)} = \frac{\rho_\Lambda(t_0)}{\rho_c(t_0)}$ (const.) $= \Omega_\Lambda^0$. Today, $\Omega_\Lambda^0 \approx 0.7$.

4) Curvature: $\Omega_K(t) = \frac{Kc^2}{H_0^2 a^2(t)} = \Omega_K^0 \frac{a_0^2}{a^2}$. Note: By definition $\Omega_K^0 = 1 - \Omega_M^0 - \Omega_R^0 - \Omega_\Lambda^0$. In principle, can by negative (open Universe) or positive (closed Universe). Today's value unknown but very close to 0 (within 2%). Must have been extremely close to 0 in the early Universe (Inflation predicts 0).

Evolution – several cases

$$\frac{da}{a}=\left(\Omega_M^0\frac{a_0^3}{a^3(t)}+\Omega_R^0\frac{a_0^4}{a^4(t)}+\Omega_\Lambda^0-\frac{Kc^2}{H_0^2a^2(t)}\right)^{1/2}H_0dt\Rightarrow\frac{da}{\left(\Omega_M^0\frac{a_0}{a(t)}+\Omega_R^0\frac{a_0^2}{a^2(t)}+\frac{a^2(t)}{a_0^2}\Omega_\Lambda^0-\frac{Kc^2}{a_0^2H_0^2}\right)^{1/2}}=a_0H_0dt$$

Matter dominance:

$$a^{1/2}da=a_0^{3/2}\sqrt{\Omega_M^0}H_0dt\Rightarrow\frac{2}{3}\left(a^{3/2}(t)-a_0^{3/2}\right)=a_0^{3/2}\sqrt{\Omega_M^0}H_0t\Rightarrow\frac{a(t)}{a_0}=\left(1+\frac{3}{2}\sqrt{\Omega_M^0}H_0t\right)^{2/3}$$

Radiation dominance:

$$ada=a_0^2\sqrt{\Omega_R^0}H_0dt\Rightarrow\frac{1}{2}\left(a^2(t)-a_0^2\right)=a_0^2\sqrt{\Omega_R^0}H_0t\Rightarrow\frac{a(t)}{a_0}=\left(1+2\sqrt{\Omega_R^0}H_0t\right)^{1/2}$$

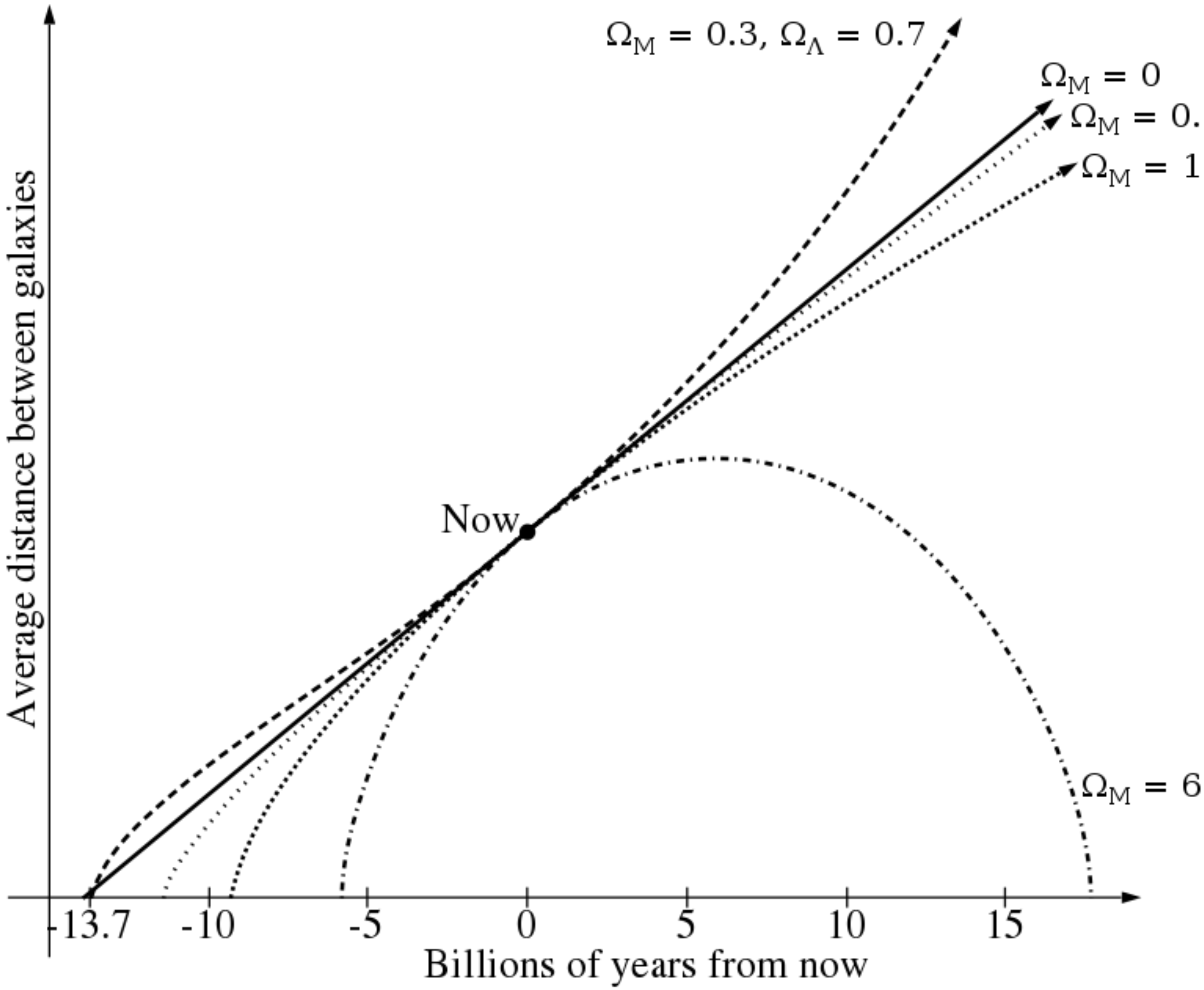
Dark energy dominance:

$$\frac{da}{a}=\sqrt{\Omega_\Lambda^0}H_0dt\Rightarrow\ln\left(\frac{a(t)}{a_0}\right)=\sqrt{\Omega_\Lambda^0}H_0t\Rightarrow\frac{a(t)}{a_0}=e^{\sqrt{\Omega_\Lambda^0}H_0t}$$

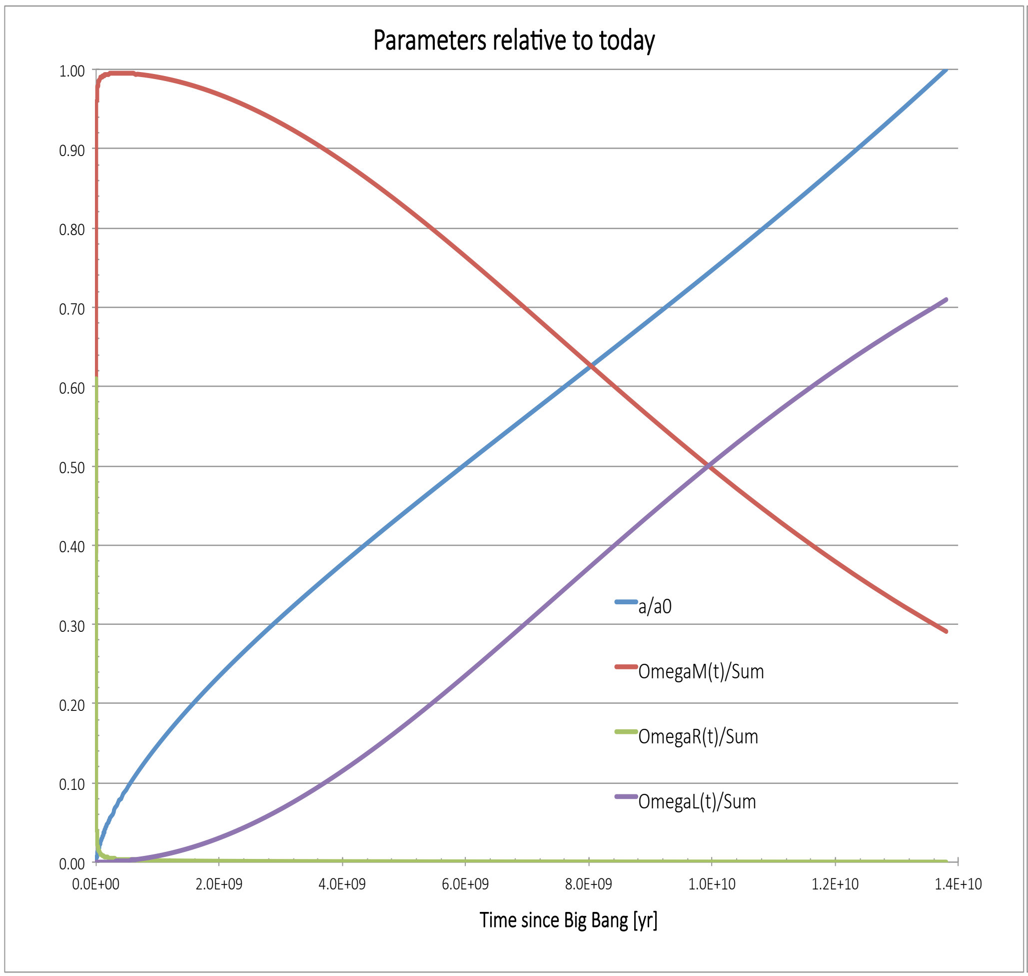
Curvature dominance (only possible if K = -1):

$$da=cdt\Rightarrow a(t)=a_0+ct$$

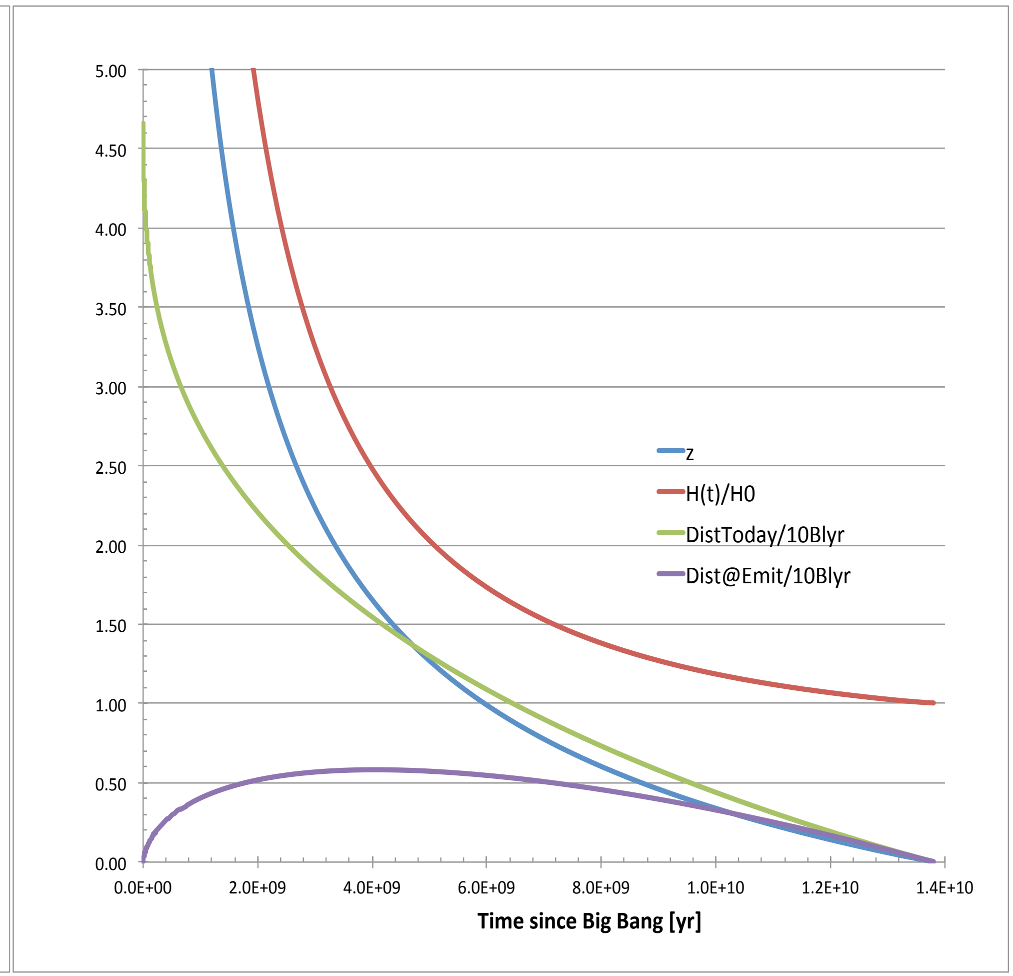
Scale factor a(t)



Consequences for our Universe



All Omegas as Fractions of Total



Distances from emitter at time *t* observed today