

Motion in a Circle

- **So far:**

Described **linear** motion of a mass point using \mathbf{x} , \mathbf{v} , \mathbf{a} , m , \mathbf{p} , \mathbf{F} .

Equations of motion: $\mathbf{a} = \Sigma \mathbf{F}/m$ or $\Delta \mathbf{p} = \Sigma \mathbf{F} \cdot \Delta t$; Equilibrium: $\Sigma \mathbf{F} = 0$

Kinetic energy $K.E. = \frac{1}{2} mv^2$; $E_{\text{tot}} = K.E. + U_{\text{pot}}$; total energy and momentum conserved in the absence of external forces

- **Now:**

We will study the motion of a single object (mass point) on a circle of radius R . We will use a new set of variables to describe this motion:

θ , ω , I , \mathbf{L} , $\boldsymbol{\tau}$

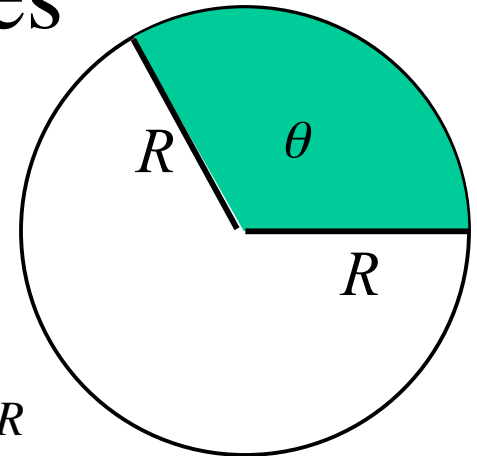
and express equations of motion and K.E. in terms of these quantities.

- **Afterwards**, we will apply these ideas to rigid bodies rotating around a fixed axis. Will find new conserved quantities and new condition for equilibrium

“New” kinematic variables

Particle going around the origin on a circle of radius R

- Use angle θ to describe position:
 - Can be measured in degrees [$^\circ$]
 - 360° is full circle
 - Circumference = distance once around the full circle = $2\pi R$
 - $\theta [^\circ] / 360 \cdot 2\pi R$ tells us by how much distance (in m) the particle has moved around the perimeter
 - \Rightarrow Can also express θ in radians [rad]: $\theta [\text{rad}] = 2\pi \cdot \theta [^\circ] / 360$
 - Distance traveled around perimeter = $R \cdot \theta$ in radians
- Angular velocity describes how fast particle goes around the circle:
 - rps = revolutions per second (1 rps = 60 RPM - “rounds per minute”)
 - If it takes time T to go all the way around once, then $1/T$ = number of rps
 - After some time Δt , particle has moved by $\Delta\theta$ [degrees] = $360^\circ \cdot rps \cdot \Delta t$;
 $\Delta\theta$ [radians] = $2\pi \cdot rps \cdot \Delta t = \omega \Delta t$; $\omega = 2\pi \cdot rps$ = angular velocity
- Linear speed $|\mathbf{v}| = 2\pi R/T = rps \cdot 2\pi R = \omega R$
 - The higher the angular velocity, the higher the linear speed
 - The further away from the center (the larger R), the higher the linear speed

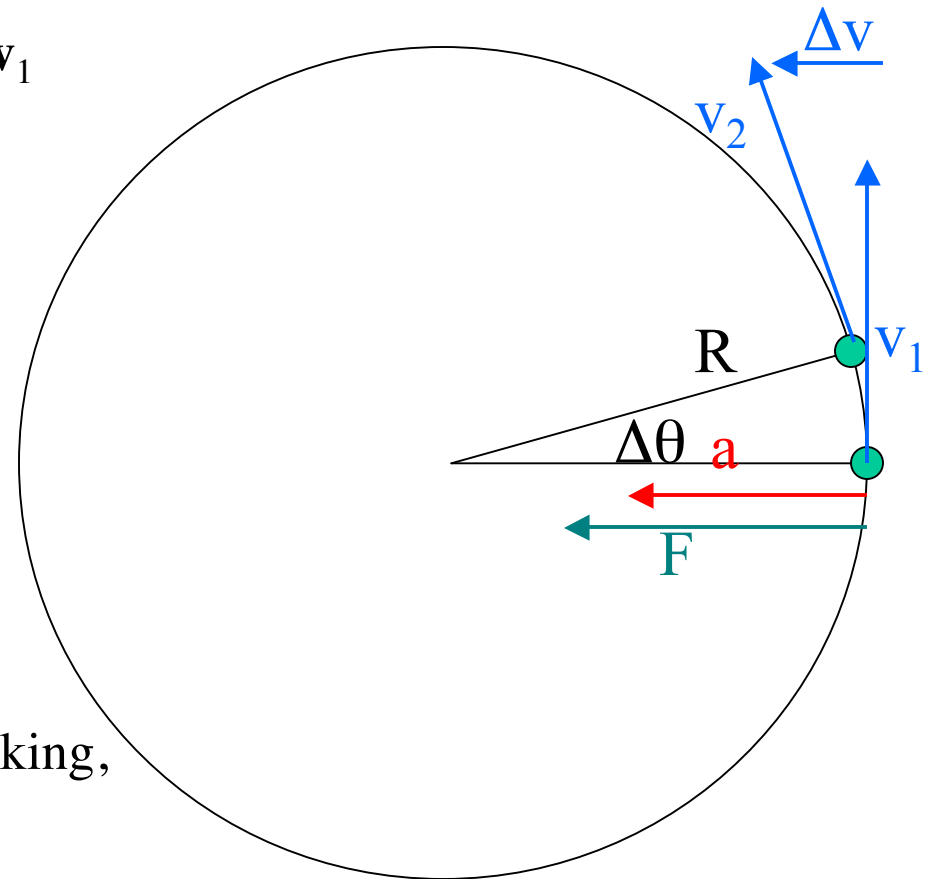


Why...

- ...do we introduce new variables?
 - Simplify description: need only **one** number for position (and **one** number for velocity); otherwise need more numbers since position, velocity are 2- or 3-dimensional vectors!
 - Can apply what we learn to rotation of extended objects (spinning wheels, cylinders, fans, blades, tops,...)
 - Will discover new conservation law (important for astronomy, ice skaters, all other rotating objects, fundamental laws of Physics):
Conservation of angular momentum **L**
 - Study new conditions for equilibrium (net torque = 0).

Something special about **circular** motion...
... it requires a (centripetal) force!
(even if you aren't speeding up or slowing down)

- After a short time Δt : $\Delta \mathbf{v} = \mathbf{v}_2 - \mathbf{v}_1$
- Change larger if $\mathbf{v}_1, \mathbf{v}_2$ larger
- Time Δt shorter if ω larger
 - $\omega = 2\pi/T, T = 2\pi R/v \Rightarrow$
- $\mathbf{a}_c = \Delta \mathbf{v} / \Delta t = v^2/R = \omega^2 R$
RADIALLY inwards
(“centripetal” acceleration)
- $\mathbf{F}_c = m\mathbf{a}_c$
(centripetal force)
- Examples:
car driving around a corner, banking,
ball on string, space station...



New dynamic variables

- Momentum? **Not** conserved ($a_{\text{rad}} \neq 0 \Rightarrow \mathbf{F}_{\text{net}} \neq 0 \Rightarrow$ forces present).
- Kinetic Energy? **Yes** conserved **if** speed (radius and angular velocity) remains constant \Rightarrow
$$\text{K.E.} = \frac{m}{2} v^2 = \frac{m}{2} (\omega \cdot R)^2 =$$
$$\frac{1}{2} (mR^2) (\omega)^2 = \frac{1}{2} I \omega^2$$
- New quantity $I = mR^2 =$ moment of inertia - all you need to know about “inertial behavior” of rotating object
- Plays the same role as mass for linear motion
- Can be used to define an analog for momentum:
 $\mathbf{L} = I \omega = mR^2 \omega = mR v =$ **angular momentum**

Angular Momentum L

- ...is another conserved quantity in Physics if no **tangential** force is acting:
 - if $R = \text{const.}$ this follows from conservation of (kinetic) energy: $\text{K.E.} = \text{const.} \Rightarrow v = \text{const.} \Rightarrow mRv = L = \text{const.}$
 - if radius R **decreases**: radial force does **positive** work $\Rightarrow v$ **increases** $\Rightarrow mRv = L = \text{const.}$
 - if radius R **increases** : radial force does **negative** work $\Rightarrow v$ **decreases** $\Rightarrow mRv = L = \text{const.}$
- ... points in the direction of axis of rotation (right hand rule:
+ = counter-clockwise rotation, - = clockwise rotation)
- Example: Ball at the end of a string: how do I , E , L , v vary with R , ω ?
- Extremely useful and important (just like conservation of \mathbf{p} and E) - see later examples with rotating objects

Now: extend to rotation of an extended object around a fixed axis

- **So far:**
We studied the motion of a **single** object (mass point) on a circle around the origin. Motion described by:
 $\theta, \omega, I, \mathbf{L}, \dots$
- **Now:** We will apply these ideas to rigid bodies rotating around a fixed axis.
- Consider extended object as a collection of (very many) mass points (atoms), each moving on a circle of radius r_p (= distance from axis).
- Obviously, each mass point has different velocity, acceleration, forces acting on it...
But: all have the **same** ω . All have the same angle θ up to a constant offset. The whole object can be described by a single I and a single L .

Kinetic energy

- Each mass point has kinetic energy
$$\text{K.E.}(P) = 1/2 (m_p r_p^2) (rps \cdot 2\pi)^2 = 1/2 I_p (\omega)^2$$
- Total kinetic energy of all points together:
add all the individual moments of inertia $I_{tot} = \sum_p (m_p r_p^2)$
- See examples next slide for specific objects;
in reality: use calculus and integrals instead of sums over lots of atoms
- $\text{K.E.}(tot) = 1/2 I_{tot} (\omega)^2$
(**Note:** there is energy in rotation!)
- I describes the whole object, just like total mass for linear motion.
Depends on **mass** and **geometrical structure** of the object and **location** and **direction** of axis!
- Can use same definition to define total angular momentum of object:
$$\mathbf{L} = \sum_p [m_p r_p^2 (\omega)] = \omega \cdot \sum_p (m_p r_p^2) = I \cdot \omega$$
 (right hand rule applies)

Moment of Inertia

- Examples:
 - Thin cylindrical shell of radius R and mass M rotating around its symmetry axis:
$$I = \sum_p (m_p r_p^2) = \sum_p (m_p R^2) = R^2 \sum_p m_p = MR^2$$
 - Solid cylinder: $I = MR^2/2$
 - Solid sphere: $I = 2/5 MR^2$
 - Thin rod of length l (axis through center): $1/12 M l^2$
 - Thin rod of length l (axis through one end): $1/3 M l^2$
 - Skinny objects rotating around their long axis have small I , extended objects or long objects rotating around their short axes have large I .
 - Objects of same overall size and mass have larger I if the mass is concentrated far away from axis (Disk race)
- Conservation of angular momentum \mathbf{L} :
 - If I increases, ω must decrease (moving mass outwards)
 - If I decreases, ω must increase (moving mass inwards)
 - Examples: ballerina, figure skating, rotating chair + person with dumbbells

Finally... - Torque!

- Tangential force times leverarm
- Plays the role of force in linear motion

- $\tau = F \cdot l$:

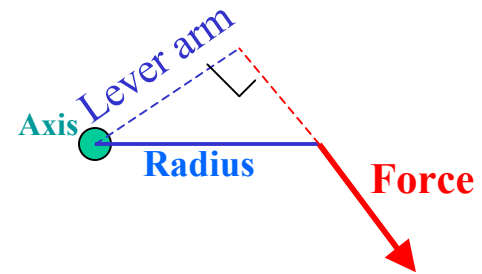
proportional to the force exerted

proportional to the length of the lever arm

only the part of the radius vector **perpendicular** to the force counts!

$$\tau = F \cdot l \sin(\theta) = \mathbf{F} \times \mathbf{l}$$

- Unbalanced torque will speed up rotational motion:
Change in angular velocity $\Delta\omega = \tau / I \Delta t$
- Unbalanced torque is the cause for any change in angular momentum:
 $\tau = \Delta L / \Delta t$



Conservation of Angular Momentum

- \mathbf{L} is a vector (pointing along axis of rotation)
- $\Delta\mathbf{L}_{\text{tot}} / \Delta t = \Sigma\boldsymbol{\tau}_P = \Sigma\boldsymbol{\tau}_{\text{external}}$ (all **internal** torques cancel assuming forces between mass points act only along connecting lines).
- If there is no net torque, then \mathbf{L} is conserved (both direction and size).
- If one part of a system changes angular momentum, another part must change by opposite amount (Bicycle wheel and rotating chair demo)
- If I changes, angular velocity must change (see earlier slide).
- K.E. is **not** conserved. Work is done by changing I (moving parts of object radially against centripetal force).
- If $\boldsymbol{\tau}$ is perpendicular to \mathbf{L} , \mathbf{L} will **only** change direction.
- Examples: Ballet, Figure Skating, in-class demos,...

New Requirements for Static Equilibrium

- **So far:**

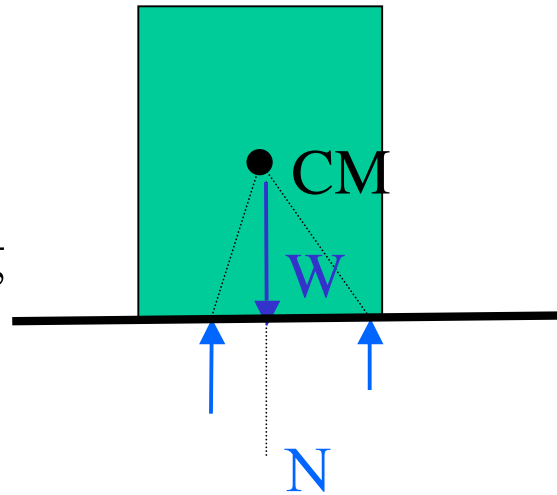
Mass points: Require $\Sigma \mathbf{F}_i = 0$ for static equilibrium (otherwise $\mathbf{a} \neq 0$). Include weight, normal forces, friction, tension in attached strings, other external forces.

- **Now:**

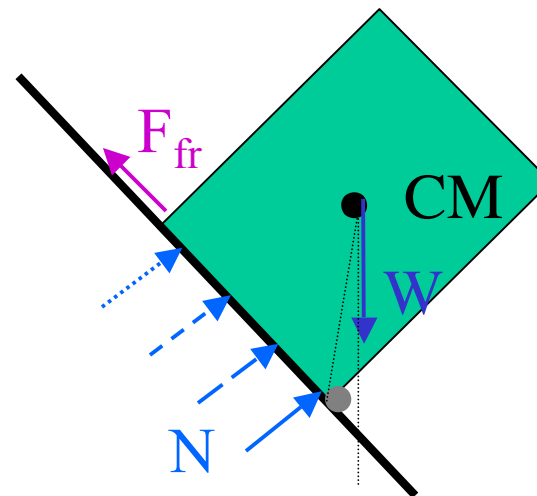
Extended objects: **Still** require $\Sigma \mathbf{F}_i = 0$. **But** : not sufficient -> if forces act on different parts of object, net torque could be non-zero => rotation.
Therefore : Require $\Sigma \boldsymbol{\tau}_i = 0$ as well.

Example I

- Center of gravity must be straight above supporting area



- Tipping over



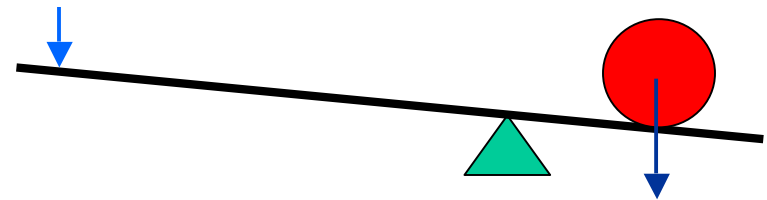
Levers and gears

- Levers: small force times large leverarm = large force times short leverarm.

- Net torque = 0
(const. ang. velocity)

- $l_F = l F$

- Same work done by either end.

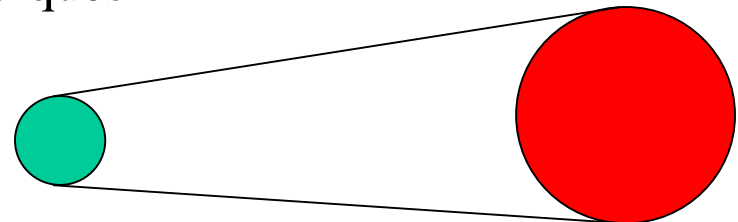


- Chains and gears

- Same Tension/force on either sprocket

- Different leverarms -> different torques

- Same work done: $\tau \Delta\Theta = T \Delta\theta$



Comparison linear motion with angular motion

- Position: $x(t)$
- Velocity: v
- Acceleration: a
- Mass: m
- Linear momentum: $p = mv$
- Force: F
- Newton's Law:
 $F = ma = \Delta p / \Delta t$
- K.E. = $m/2 v^2$
- Momentum conserved if $\Sigma F = 0$
(no net force)
- Change of K.E.:
 $\Delta \text{K.E.} = \Delta W = F \Delta x$
- Angular Position: θ
- Angular velocity: rps, ω
- Angular acceleration: α
- Moment of Inertia: $I = mR^2$
- Angular Momentum: $L = I \omega$
- Torque: $\tau = R F_{\text{tan}}$
- "Newton's" Law:
 $\tau = I\alpha = \Delta L / \Delta t$
- K.E. = $1/2 I (2\pi rps)^2 = 1/2 I \omega^2$
- L conserved always if $\Sigma \tau = 0$
(no net torque)
- Change of K.E.:
 $\Delta W = \tau \Delta \theta$

Summary: Motion is in 2D,
but can be described by
single (1D) variables