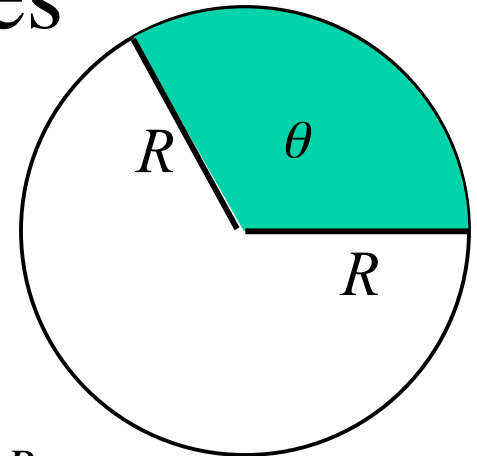


Motion in a Circle

- **So far:**
Described **linear** motion of a mass point using \mathbf{x} , \mathbf{v} , \mathbf{a} , m , \mathbf{p} , \mathbf{F} .
Equations of motion: $\mathbf{a} = \mathbf{F}/m$; Kinetic energy K.E. = $\frac{1}{2} mv^2$
- **Now:**
We will study the motion of a single object (mass point) on a circle of radius R . We will use a new set of variables to describe this motion:
 $\theta, \omega, \alpha, I, \mathbf{L}, \boldsymbol{\tau}$
and express equations of motion and K.E. in terms of these quantities.
- **Afterwards** , we will apply these ideas to rigid bodies rotating around a fixed axis.

“New” kinematic variables



Particle going around the origin on a circle of radius R

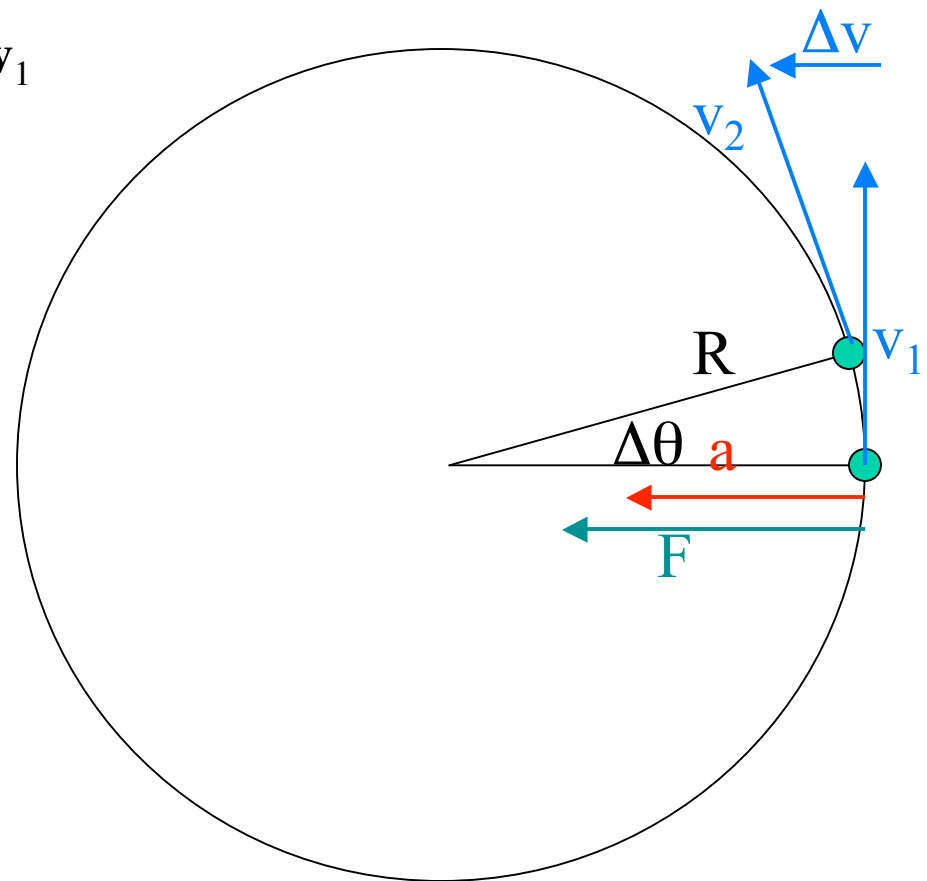
- Use angle θ to describe position:
 - Can be measured in degrees [$^\circ$]
 - $\theta/360^\circ$ is fraction of full circle (FFC)
 - Circumference = distance once around the full circle = $2\pi R$
 - $FFC \cdot 2\pi R$ tells us by how much the particle has moved along the perimeter
 - \Rightarrow Can also express θ in radians [rad]: $FFC \cdot 2\pi = 2\pi \cdot \theta/360^\circ$
 - Distance traveled around perimeter = $R \cdot \theta$ in radians
- Angular velocity describes how fast particle goes around the circle:
 - rps = revolutions per second (1rps = 60 RPM - “rounds per minute”)
 - If it takes time T to go all the way around once, then $1/T = rps$
 - After some time Δt , particle has moved by $FFC = rps \cdot \Delta t \Rightarrow$
 $\Delta\theta$ [degrees] = $360^\circ \cdot rps \cdot \Delta t$; $\Delta\theta$ [radians] = $2\pi rps \cdot \Delta t = \omega \Delta t$
- Linear speed $|v| = 2\pi R/T = rps \cdot 2\pi R = \omega R$ (Note: $rps \cdot 2\pi = \omega$)
 - The higher the angular velocity, the higher the linear speed
 - The further away from the center (the larger R), the higher the linear speed

Why...

- ...do we introduce new variables?
 - Simplify description: need only **one** number for position and **one** more for velocity (compared with usual position, velocity = vectors!)
 - Can apply what we learn to rotation of extended objects (spinning wheels, cylinders, fans, blades, tops,...)
 - Will discover new conservation law (important for astronomy, ice skaters, many other rotating objects, fundamental laws of Physics): Conservation of angular momentum **L**
 - Study new conditions for equilibrium (net torque = 0).

Something special about **circular** motion...
... it requires a (centripetal) force!
(even if you aren't speeding up or slowing down)

- After a short time Δt : $\Delta \mathbf{v} = \mathbf{v}_2 - \mathbf{v}_1$
- Change larger if $\mathbf{v}_1, \mathbf{v}_2$ larger
- Time Δt shorter if ω larger
 - $\omega = 2\pi/T, T = 2\pi R/v \Rightarrow$
- $\mathbf{a}_c = \Delta \mathbf{v} / \Delta t = v^2/R$
RADIALLY inwards
("centripetal")
- $\mathbf{F}_c = m\mathbf{a}_c$
- Examples:
car driving around a corner,
banking, ball on string,...



New dynamic variables

- Momentum? **Not** conserved ($a_{\text{rad}} \neq 0 \Rightarrow \mathbf{F}_{\text{net}} \neq 0 \Rightarrow$ external forces present).
- Kinetic Energy? **Yes** conserved **if** speed (radius and angular velocity) remains constant \Rightarrow
K.E. = $\frac{m}{2} v^2 = \frac{m}{2} (\omega \cdot R)^2 =$
 $\frac{1}{2} (mR^2) (\omega)^2 = \frac{1}{2} I \omega^2$
- New quantity $I = mR^2 =$ moment of inertia
- Plays the same role as mass for linear motion
- Can be used to define an analog for momentum:
 $L = I (\omega) = mR^2 \omega = mR v =$ **angular momentum**
- Example: Ball at the end of a string: how do I, E, L vary with R, ω ?

Now: extend to rotation of an extended object around a fixed axis

- **So far:**
We studied the motion of a **single** object (mass point) on a circle around the origin. Motion described by:
 θ , rps , I , \mathbf{L} , ...
- **Now:** We will apply these ideas to rigid bodies rotating around a fixed axis.
- Consider extended object as a collection of (very many) mass points, each moving on a circle of radius r_p (= distance from axis).
- Obviously, each mass point has different velocity, acceleration, forces acting on it...
But: all have the **same** ω . All have the same angle θ up to a constant offset. The whole object can be described by a single I and a single L .

Kinetic energy

- Each mass point has kinetic energy
$$\text{K.E.}(P) = 1/2 (m_P r_P^2) (rps \cdot 2\pi)^2 = 1/2 I_P (rps \cdot 2\pi)^2$$
- Total kinetic energy of all points together:
add all the individual moments of inertia $I_{tot} = \sum_p (m_P r_P^2)$
- $\text{K.E.}(tot) = 1/2 I_{tot} (rps \cdot 2\pi)^2$
- (**Note:** there is energy in rotation!)
- Moment of inertia for an extended object:
$$I = \sum_p (m_P r_P^2)$$
- Describes the whole object, just like total mass for linear motion.
Depends on **mass** and **geometrical structure** of the object and **location** and **direction** of axis!

Moment of Inertia

- Examples:
 - Thin cylindrical shell of radius R and mass M rotating around its symmetry axis:
$$I = \sum_p (m_p r_p^2) = \sum_p (m_p R^2) = R^2 \sum_p m_p = MR^2$$
 - Solid cylinder: $I = MR^2/2$
 - Solid sphere: $I = 2/5 MR^2$
 - Thin rod (axis through center): $1/12 ML^2$
 - Thin rod (axis through one end): $1/3 ML^2$
 - In general: skinny objects rotating around their long axis have small I , extended objects or long objects rotating around their short axes have large I .
 - Objects of same overall size and mass have larger I if the mass is concentrated far away from axis (Disk race)
- Can use same definition to define total angular momentum of object:
$$L = \sum_p [m_p r_p^2 (rps \cdot 2\pi)] = (rps \cdot 2\pi) \sum_p (m_p r_p^2) = I (rps \cdot 2\pi)$$

Finally... - Torque!

- Twisting action
- Plays the role of force in linear motion
- Due to a force exerted with a lever arm

- $\tau = F \cdot l$:

proportional to the force exerted

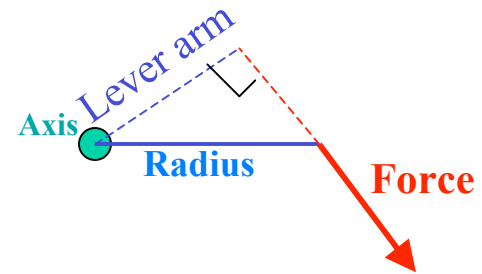
proportional to the length of the lever arm

only the part of the radius vector **perpendicular** to the force counts!

$$\tau = F \cdot l \sin(\theta) = F \times l$$

- Unbalanced torque will speed up rotational motion:
Change in angular velocity $\Delta\omega = \alpha \Delta t = \tau/I \Delta t$
- Unbalanced torque is the cause for any change in angular momentum:

$$\tau = \Delta L / \Delta t$$



Conservation of Angular Momentum

- \mathbf{L} is a vector (pointing along axis of rotation)
- $d\mathbf{L}_{\text{tot}}/dt = \Sigma\boldsymbol{\tau}_P = \Sigma\boldsymbol{\tau}_{\text{external}}$ (all **internal** torques cancel **if** forces between mass points act only along connecting lines).
- If there is no net torque, then \mathbf{L} is conserved (both direction and size).
- If I decreases, angular velocity must increase!
- K.E. is **not** conserved. Work is done by changing I (moving parts of object radially against centripetal force).
- If $\boldsymbol{\tau} \perp \mathbf{L}$, \mathbf{L} will **only** change direction.
- Examples: Ballet, Figure Skating, in-class demos,...

New Requirements for Static Equilibrium

- **So far:**

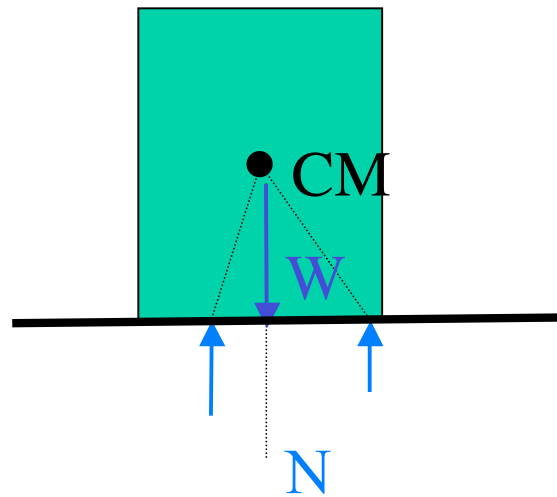
Mass points: Require $\Sigma \mathbf{F}_i = 0$ for static equilibrium (otherwise $\mathbf{a} \neq 0$). Include weight, normal forces, friction, tension in attached strings, other external forces.

- **Now:**

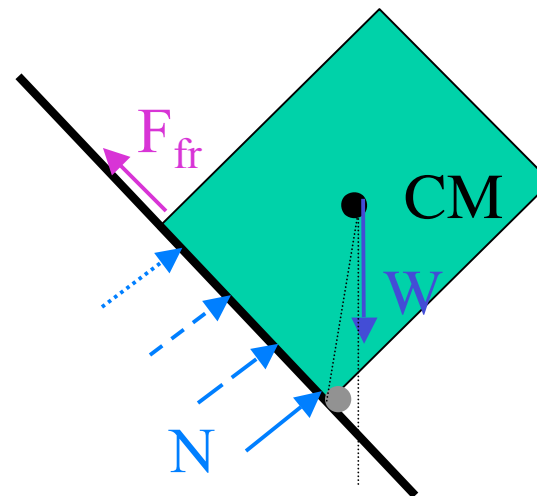
Extended objects: **Still** require $\Sigma \mathbf{F}_i = 0$. **But** : not sufficient -> if forces act on different parts of object, net torque could be non-zero => rotation.
Therefore : Require $\Sigma \boldsymbol{\tau}_i = 0$ as well.

Example I

- Center of gravity vs. support

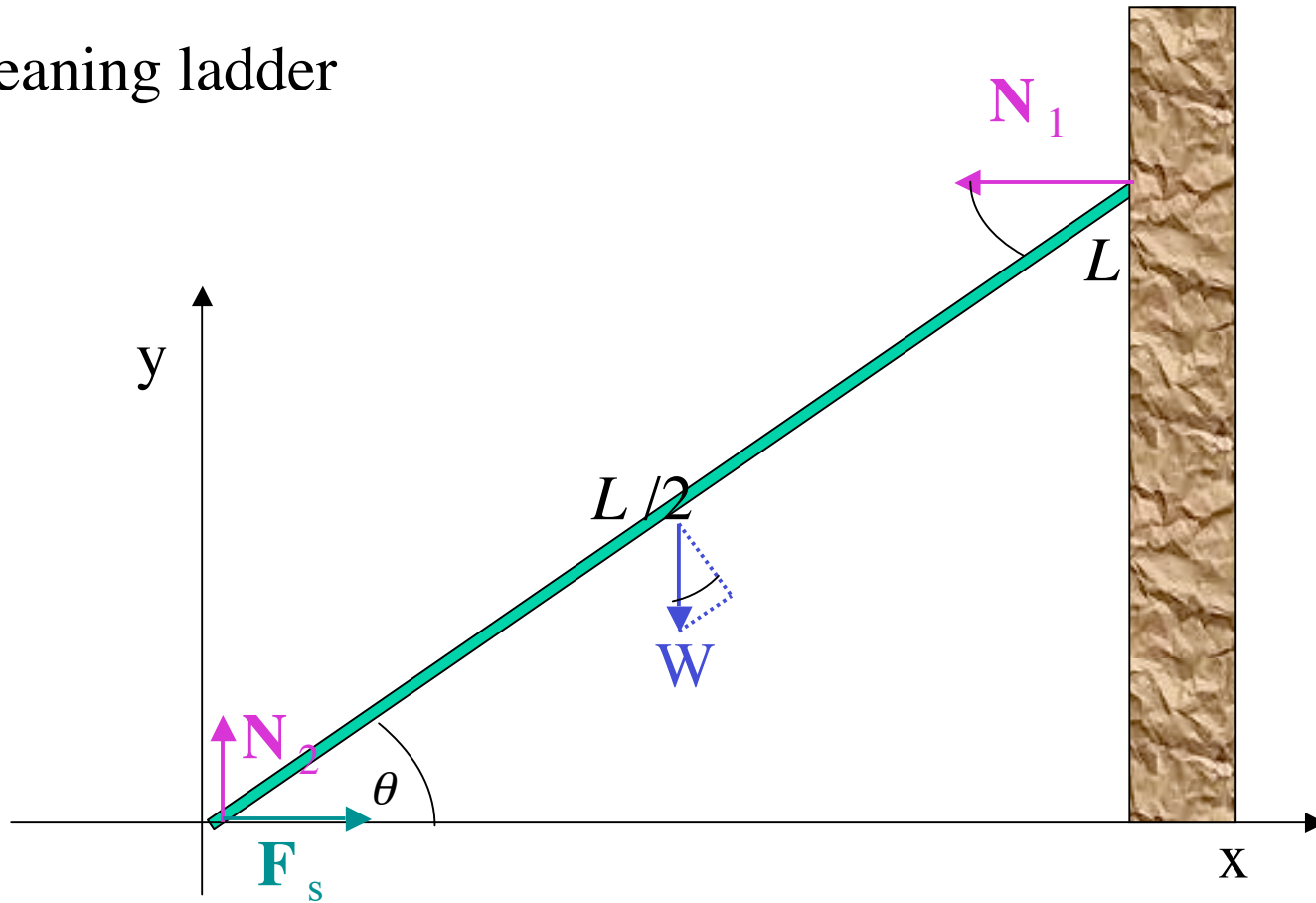


- Tipping over



Example II

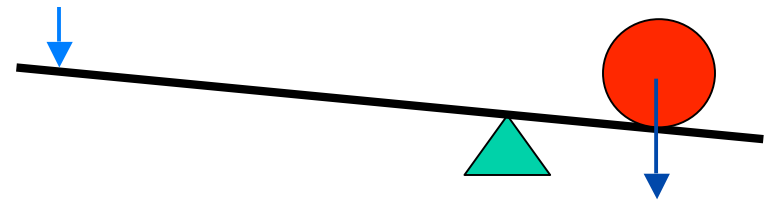
- Leaning ladder



Levers and gears

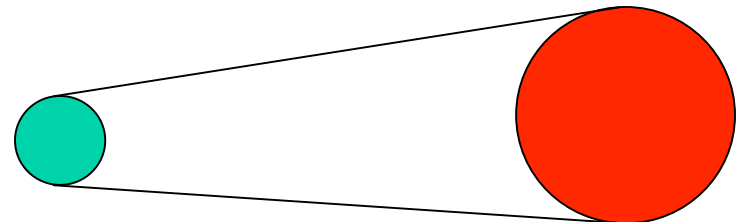
- Levers: small force times large leverarm = large force times short leverarm.

- Net torque = 0
(const. ang. velocity)
- $L f = l F$
- Same work done by either end.



- Chains and gears

- Same Tension/force on either sprocket
- Different leverarms -> different torques
- Same work done: $\tau \Delta\Theta = T \Delta\theta$



Comparison linear motion with angular motion

- Position: $x(t)$
- Velocity: v
- Acceleration: a
- Mass: m
- Linear momentum: $p = mv$
- Force: F
- Newton's Law:
 $F = ma = dp/dt$
- K.E. = $m/2 v^2$
- Momentum conserved if $\Sigma F = 0$
(no net force)
- Change of K.E.:
 $\Delta \text{K.E.} = \Delta W = F \Delta x$
- Angular Position: θ
- Angular velocity: rps, ω
- Angular acceleration: α
- Moment of Inertia: $I = mR^2$
- Angular Momentum: $L = I \omega$
- Torque: $\tau = R F_{\text{tan}}$
- "Newton's" Law:
 $\tau = I\alpha = dL/dt$
- K.E. = $1/2 I (2\pi rps)^2 = 1/2 I \omega^2$
- L conserved always if $\Sigma \tau = 0$
(no net torque)
- Change of K.E.:
 $\Delta W = \tau \Delta \theta$

Summary: Motion is in 2D,
but can be described by
single (1D) variables