Image Processing

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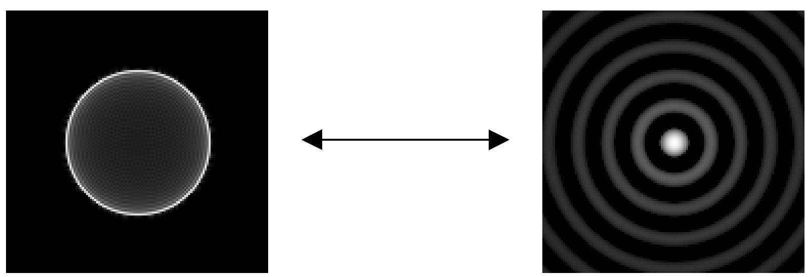
2-D FT preliminaries and properties

Introduction

In one dimension the FFT of a dataset of length N requires (of order) Nlog2N operations. In two dimensions an array of MxN requires (of order) MNlog2MN operations for the FFT compared to MN(M+N) for the DFT. For an array of 1024x1024 this represents a very useful speed increase factor of 50,000.



Properties of the 2D Fourier Transform

Definitions

$$F(u,v) = \iint f(x,y)\,\mathbf{exp}\,(-2\pi i[ux + vy])dxdy$$

Continuous inverse Fourier transform:

$$f(x,y) = \iint F(u,v)\,\mathbf{exp}\,(+2\pi i[ux + vy])dudv$$

Discrete Fourier transform:

$$F(m,n) = \sum_{l=-N/2}^{N/2-1} \sum_{k=-M/2}^{M/2-1} f(k,l)\,\mathbf{exp}(-2\pi i[mk / M + nl / N])$$

Discrete inverse Fourier transform:

$$f(m,n) = \frac{1}{MN} \sum_{l=-M/2}^{M/2-1} \sum_{k=-M/2}^{M/2-1} F(k,l)\,\mathbf{exp}(+2\pi i[mk / M + nl / N])$$

It should be noted that the forward and inverse transforms above can be defined in a more symmetric fashion by using a normalization constant of $1/\sqrt{MN}$. The range of values used are $-\frac{N}{2} \le n < \frac{N}{2}$ and $-\frac{M}{2} \le m < \frac{M}{2}$

Consequently the zero frequency pixel appears slightly offset from the exact centre of the array. Continuous FTs are used in the theory that follows. It is a useful exercise for the student to rewrite the equations for the corresponding discrete transform in each case.

Implementation

The 2D FT can be implemented as two consecutive 1D FTs: first in the x direction, then in the y direction (or vice versa). Symbolically:

$$F_1(u,y) = \int f(x,y)\,\mathbf{exp}(-2\pi i[ux])dx$$
$$F(u,v) = \int F_1(u,y)\,\mathbf{exp}(-2\pi i[vy])dy$$

The software provided uses the 1D FFT code (FOUR1) originating from Numerical Recipes. On the first pass all the rows are transformed. On the second pass columns are transformed. The same method of splitting into sequential 1D FTs can be used for higher dimensions, but the method is not optimal in terms of memory utilized and has been surpassed by more sophisticated algorithms.

Exercises Using 2D FFT's

The exercises in the next four modules over the next four weeks are intended to give students an intuitive feel for 2D Fourier transforms. The physical (in this case visual) interpretation of Fourier theorems is often missing in the more symbolic approach typical of mathematics departments. Although many of the exercises are qualitative in nature it should not be concluded that the 2D FFT is not an important quantitative tool.

Module 1: Fundamental Properties of the 2D FT

A) Shift Property

This property is just a simple extension of the 1D theory:

$$f(x-x_0, y-y_0) \leftrightarrow F(u,v)\,\mathbf{exp}\,(-2\pi i[ux_0 + vy_0])$$

The notation $\leftrightarrow$ indicates a Fourier transform pair, as before. In other words, the FT of a shifted function is unaltered except for a linearly varying phase factor. The actual phase ϕ can be thought of as an inclined plane $\phi = -2\pi i[ux_0 + vy_0]$ intersecting the origin.

- Write down the FT of a shifted delta function $\delta(x-x_0)\delta(y-y_0)$
- Consider two images which differ only in the location of the central white spot. By viewing the FFTs of both these function estimate the relative x and y shift components. HINT: Consider the number of fringes in each of the FT phase displays, and resolve the fringes into x and y components.
- How might it be possible to shift a function by 1/100th of a pixel without changing the modulus of its FT?

B) Scaling Property

This property is best summarized by "a contraction in one domain produces corresponding expansion in the Fourier domain".

$$f(ax,by) \leftrightarrow \frac{1}{|ab|} F\left(\frac{u}{a},\frac{v}{b}\right)$$

- Is the scaling property exact for the discrete FT ?
- Carefully look at the phase component of psd of an image. What values of phase are present and what is their distribution?
- Make use of the ZOOM function to examine regions of interest.

C) Rotation Property

This is a property which has no analogy in one dimension.

$$\left.\begin{array}{l} f(x,y) \leftrightarrow F(u,v) \\ f(x',y') \leftrightarrow F(u',v') \end{array}\right\}$$

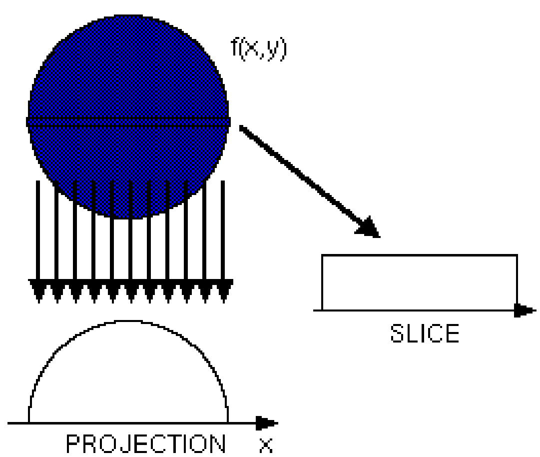
$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

- Simply stated: if a function is rotated, then its Fourier transform rotates an equal amount.

D) Projection-Slice Theorem

Again this is a property which has no useful analogy in one dimension. The theorem can be stated briefly: the (1D) Fourier transform of the projection of a 2D function is the central slice of the Fourier transform of that function.



If F(u,0) is the central slice, and $\left[\int f(x,y)dy\right]$ defines the projection of f(x,y) on the x axis then:

$$F(u,0) = \int \left[\int f(x,y)dy\right] \exp(-2\pi i u x)dx$$

- The above result can be generalised to slices and projections in any direction, not just along the x or y axes. The projection-slice theorem can also be generalised to higher dimensions and is the basis of CAT (computer aided tomography) reconstruction of 3D images from 1D and 2D sections.
- Comment on the FT of an image that has diffraction-like fringes.

E) Fourier Transform of Circularly Symmetric Functions

This is an important aspect of Fourier theory, especially for diffraction in optical systems which are often symmetric about the optical axis. In such systems the 2D FT can be reduced to a 1D tranform. The 1D transform in this case is known as the Hankel tranform. Unfortunately there is no equivalent to the FFT for the discrete Hankel transform. It should be noted that, by using the projection-slice property, the Hankel tranform of a radial (slice) function is equivalent to the Fourier transform of the projection of a radial function. The derivation is as
follows:

$$f(x,y) = f(r) \qquad x^2 + y^2 = r^2$$
$$F(u,v) = F(q) \qquad u^2 + v^2 = q^2$$

Use coordinate transforms:

$$x = r\cos\theta \qquad u = q\cos\phi$$
$$y = r\sin\theta \qquad v = q\sin\phi$$

Then

$$F(u,v) = \int_0^\infty f(r)\left\{\int_0^{2\pi} \exp(-2\pi i q r\cos[\theta-\phi])d\theta\right\} r dr$$

The term in curly brackets is the zero order Bessel function, hence

$$F(q) = 2\pi\int_0^\infty f(r)\left\{J_0(2\pi q r)\right\} r dr$$

F(q) is the Hankel transform of f(r).

- Now reconsider the Einstein image. Obtain f(r) in this case?
- What is the 2D FT of a separable function f(x,y)=g(x).h(y) in terms of the 1D FTs G(u) and H(v) ?
- By writing the function in separable form find the 2D FT for circular Gaussian:

HINT:
$$f(r) = \exp\left(-\pi\left[\frac{x^2+y^2}{a^2}\right]\right)$$

It is now possible to state the Hankel transform of a Gaussian:

$$f(r) = \exp\left(-\pi\left[\frac{r^2}{a^2}\right]\right)$$

- Write down the expression for F(q).
- Note that a circular Gaussian is both circularly symmetric and separable.

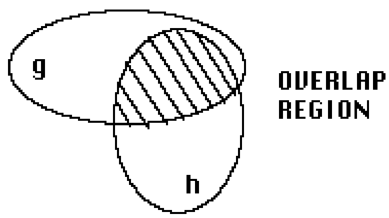
Convolution in Two Dimensions

You may remember covering the concept of convolution in Chapter_3, Module_4 of this course. Make sure you are acquainted with the one dimensional convolution therein. The mathematical treatment is easily extended from one dimension to two dimensions:

$$f(x,y) = g ** h = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x',y')h(x-x',y-y')dx'dy'$$

The symbol ** is used here to represent full convolution in two dimensions. Note that for each value of x and y the integral has to be evaluated over all values of x' and y'.

Such an operation can be thought of as an overlap intergral. The functions g and h are overlapped with a certain displacement between them and the total "weight" of the overlapping region is evaluated. The displacement is changed and the overlap re-evaluated. The process is repeated for all possible displacements.



For the discretely sampled datasets we are considering in this course such a summation would require a large number of computations (of the order M2N2). The Fourier transform of this
convolution is given by F(u,v), where:

$$F(u,v) = G(u,v)H(u,v)$$

Alternatively this may be written using the Fourier operator F in a particularly simple and useful form:

$$\mathbf{F}\ \{g ** h\} = \mathsf{F}\ \{g\}\mathsf{F}\ \{h\}$$

The Fourier transform of a convolution is the product of the Fourier transforms of the functions.

Using the Fourier transform to evaluate a convolution requires of the order MNlog(M+N) operations, which is a considerable improvement over the simple summation method.