THE PATH-INTEGRAL FORMALISM OF QUANTUM MECHANICS

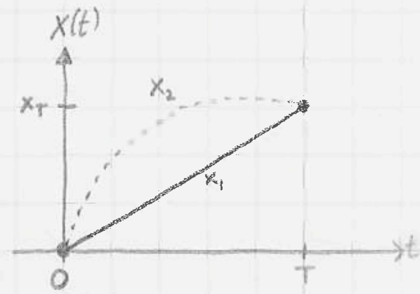
Before introducing this formalism, let's remind ourselves of how the extemisation of action enters in classical mechanics, "Hamilton's principle".

* The path of a classical particle is that which extremises (usually minimises) the action, usually defined by $S(x,\dot{x}) = \int_{t_i}^{t_f} dt\, L(x,\dot{x})$, $L(x,\dot{x}) = \frac{1}{2}m\dot{x}^2 - V(x)$

e.g. a free-particle in one-dimension, $L = \frac{1}{2}m\dot{x}^2$

with initial condition, $x(0) = 0$

& final condition $x(T) = x_T$



we know the answer is $x_1(t) = \left(\frac{x_T}{T}\right)t$

$\dot{x}_1 = x_T/T \Rightarrow L = \frac{1}{2}m\, x_T^2/T^2$

& $S = \int_0^T dt \cdot \frac{1}{2}m\, x_T^2/T^2 = \underline{\frac{1}{2}m x_T^2/T}$

suppose we evaluate the action for a path x_2 : $x_2(t) = at + bt^2$

$x_2(T) = x_T = aT + bT^2 \quad \Rightarrow x_2(t) = \left(\frac{x_T - bT^2}{T}\right)t + bt^2$

$\dot{x}_2(t) = \frac{x_T - bT^2}{T} + 2bt \quad \Rightarrow (\dot{x}_2(t))^2 = \left(\frac{x_T}{T} - bT + 2bt\right)^2$

& $S = \int_0^T dt\, \frac{1}{2}m\dot{x}_2^2 = \frac{1}{2}m x_T^2/T + \frac{1}{3}b^2T^3 > \frac{1}{2}m\, x_T^2/T$

all paths except the straight line will give a larger action than $\frac{1}{2}m\, x_T^2/T$.

We'd like to consider the way in which action enters into quantum mechanics - this is most easily done in terms of the "propagator". (We considered this in problem set 4 last semester)

The propagator (in position space) 'evolves' a wavefunction at time t to a time, t' :

$$\psi(x,t) = \int dx'\, K(x-x', t-t')\, \psi(x',t') \quad , \quad \boxed{\begin{aligned} K(x-x', t-t') &= \langle x|\hat{U}(t,t')|x'\rangle \\ \hat{U}(t,t') &= e^{-i\hat{H}(t-t')/\hbar} \end{aligned}}$$

e.g. a free-particle $\hat{H} = \hat{P}^2/2m$

$$K(x,t) = \langle x|e^{-i\hat{p}^2/2m \cdot t/\hbar}|0\rangle = \int dp\, \langle x|e^{-i\hat{p}^2/2m\, t/\hbar}|p\rangle\langle p|0\rangle$$

$$= \int dp\, e^{-ip^2/2m\, t/\hbar}\langle x|p\rangle\langle p|0\rangle = \int dp\, \exp\left(-i\frac{p^2}{2m}\frac{t}{\hbar} + ipx/\hbar\right)\frac{1}{\sqrt{2\pi\hbar}}$$

$$\boxed{K(x,t) = \left(\frac{m}{2\pi i\hbar t}\right)^{1/2} e^{i\frac{mx^2}{2\hbar t}}}$$

The mathematically correct object to consider in the <u>regulated</u>, <u>causal</u> propagator

$$K_\eta(x_b,t_b\,;x_a,t_a) \equiv \Theta(t_b-t_a)\,\langle x_b|\exp\left[-i\left(\hat H(\hat p,\hat x)-i\eta\right)/\hbar\,(t_b-t_a)\right]|x_a\rangle$$

from here we'll not explicitly write the step fn or the regulator, but they should be there.

Suppose we subdivide the time-interval t_b-t_a into $N+1$ segments of length ε:

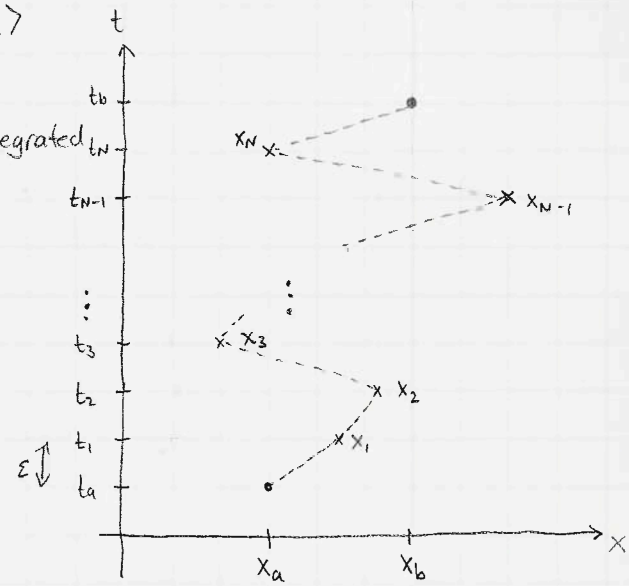
$$t_0=t_a \quad ; \quad t_n = t_a+\varepsilon\, n \quad ; \quad t_b = t_{N+1} = t_a+\varepsilon(N+1)$$

then $$e^{-i\hat H(t_b-t_a)} = e^{-i\hat H(t_b-t_N)}\,e^{-i\hat H(t_N-t_{N-1})}\,e^{-i\hat H(t_{N-1}-t_{N-2})}\cdots e^{-i\hat H(t_1-t_a)}$$

We can insert a complete set of position eigenstates between each exponential $\quad 1=\int dx_n\,|x_n\rangle\langle x_n|$

$$K(x_b,t_b\,;x_a,t_a) = \int dx_1 dx_2\cdots dx_N\,\langle x_b|e^{-i\hat H(t_b-t_N)/\hbar}|x_N\rangle\langle x_N|e^{-i\hat H(t_N-t_{N-1})/\hbar}|x_{N-1}\rangle\langle x_{N-1}|e^{-i\hat H(t_{N-1}-t_{N-2})/\hbar}|x_{N-2}\rangle\cdots$$
$$\cdots\langle x_1|e^{-i\hat H(t_1-t_a)/\hbar}|x_a\rangle$$

notice that this is the product of propagators integrated over all intermediate positions



clearly we are <u>integrating</u> <u>over</u> <u>all</u> <u>possible</u> <u>paths</u> between x_a at time t_a and x_b at time t_b.

But we've cheated by making time discrete, we need to take the limit that makes time continuous.

In the limit of small ε: $\quad \langle x_n|e^{i\hat H\varepsilon/\hbar}|x_{n-1}\rangle \to \langle x_n|1-i\hat H\varepsilon/\hbar|x_{n-1}\rangle$

$$= \langle x_n|x_{n-1}\rangle - i\varepsilon/\hbar\langle x_n|\hat H|x_{n-1}\rangle$$
$$= \delta(x_n-x_{n-1}) - i\varepsilon/\hbar\langle x_n|\hat H|x_{n-1}\rangle$$

consider working with a variable corresponding to the conjugate momentum, p:

$$\delta(x_n-x_{n-1}) = \int\frac{dp}{2\pi\hbar}\,e^{ip(x_n-x_{n-1})/\hbar}$$

$$\langle x_n|\hat p^2|x_{n-1}\rangle = \int\frac{dp}{2\pi\hbar}\,p^2\,e^{ip(x_n-x_{n-1})/\hbar}$$

$$\langle x_n|V(\hat x)|x_{n-1}\rangle = V(x_n)\,\delta(x_n-x_{n-1}) \quad \text{for a local pot}^{\underline{l}}$$
$$= \int\frac{dp}{2\pi\hbar}\,V(x_n)\,e^{ip(x_n-x_{n-1})/\hbar}$$

$$\langle x_n|\hat H|x_{n-1}\rangle = \int\frac{dp}{2\pi\hbar}\left(\frac{p^2}{2m}+V(x_n)\right)e^{ip(x_n-x_{n-1})/\hbar}$$

$$\langle x_n | e^{-i\hat{H}\varepsilon/\hbar} | x_{n-1}\rangle = \int \frac{dp}{2\pi\hbar} e^{ip(x_n-x_{n-1})/\hbar} \left(1 - \frac{i\varepsilon}{\hbar}\left(\frac{p^2}{2m} + V(x_n)\right)\right) = \int \frac{dp}{2\pi\hbar} e^{ip(x_n-x_{n-1})/\hbar} \Big(1 - \frac{i\varepsilon}{\hbar} \underbrace{H(p,x_n)}_{\text{an ordinary function}}\Big)$$

$$\simeq \int \frac{dp}{2\pi\hbar} \exp\left(ip(x_n-x_{n-1})/\hbar - \frac{i\varepsilon}{\hbar} H(p,x_n)\right)$$

$$= \int \frac{dp}{2\pi\hbar} \exp\left(\frac{i\varepsilon}{\hbar}\left[p\,\frac{x_n-x_{n-1}}{\varepsilon} - H(p,x_n)\right]\right) \qquad \text{define } \frac{x_n-x_{n-1}}{\varepsilon} = \frac{x_n-x_{n-1}}{t_n-t_{n-1}} = \dot{x}_n$$

$$= \int \frac{dp}{2\pi\hbar} e^{i\varepsilon[p\dot{x}_n - H(p,x_n)]/\hbar}$$

thus $K(x_b t_b ; x_a t_a) = \int dx_1 dx_2 \cdots dx_N \int \frac{dp_1}{2\pi\hbar}\frac{dp_2}{2\pi\hbar}\cdots\frac{dp_{N+1}}{2\pi\hbar} \exp\left[\frac{i\varepsilon}{\hbar}\sum_{n=1}^{N+1}\left(p_n\dot{x}_n - H(p_n,x_n)\right)\right]$

we can perform the integrals over p_n in the case that $H(p,x) = \frac{p^2}{2m} + V(x)$:

$$\int \frac{dp_n}{2\pi\hbar} \exp\left(\frac{i\varepsilon}{\hbar}\left(p_n\dot{x}_n - p_n^2/2m\right)\right) = \frac{1}{2\pi\hbar}\int dp_n \exp\left(-\frac{i\varepsilon}{2m\hbar}\left(p_n^2 - 2m\dot{x}_n p_n\right)\right)$$

$$= \frac{1}{2\pi\hbar}\int dp_n \exp\left(\frac{-i\varepsilon}{2\hbar m}(p_n - m\dot{x}_n)^2\right)\exp\left(\frac{+i\varepsilon}{2\hbar m} m^2\dot{x}_n^2\right) = \frac{1}{2\pi\hbar} e^{i\varepsilon m\dot{x}_n^2/2\hbar}\int_{-\infty}^{\infty} dy\, e^{-\beta^2 y^2} \qquad \left[\beta = \sqrt{\frac{i\varepsilon}{2\hbar m}}\right]$$

$$= \frac{1}{2\pi\hbar}\frac{\sqrt{\pi}}{\beta}\cdot e^{i\varepsilon m\dot{x}_n^2/2\hbar} = \sqrt{\frac{m}{2\pi\hbar i\varepsilon}}\, e^{i\varepsilon m\dot{x}_n^2/2\hbar}$$

[* of course these oscillatory integrals only converge because of the $e^{-\eta/\hbar}$ factor we've not been writing *]

$$\Rightarrow K(x_b t_b ; x_a t_a) = \left(\frac{m}{2\pi\hbar i\varepsilon}\right)^{(N+1)/2}\int dx_1\cdots dx_N \exp\left(\frac{i\varepsilon}{\hbar}\sum_{n=1}^{N+1}\left(\tfrac{1}{2}m\dot{x}_n^2 - V(x_n)\right)\right)$$

limit $\varepsilon \to 0$, $N \to \infty$ with $t_b - t_a = (N+1)\varepsilon$ = constant

$$\longrightarrow \int \mathcal{D}x \exp\left[\frac{i}{\hbar}\int_{t_a}^{t_b} dt\, L(x,\dot{x})\right] = \int \mathcal{D}x\, e^{iS[x]/\hbar}$$

$\int \mathcal{D}x$ = <u>functional</u> <u>integral</u> over all possible paths between x_a @ t_a & x_b @ t_b

Notice the major difference w.r.t classical physics – <u>all paths</u> are "allowed", not just the one with minimum action, each one contributes to the propagator with a weight $e^{iS[x]/\hbar}$.

e.g. the quantum mechanical free-particle, $V(x) = 0$.

$$S[x] = \int_{t_a}^{t_b} dt \cdot \tfrac{1}{2} m \dot{x}^2 \qquad x(t) = \text{any path with endpoints} \quad x(t_a) = x_a, \ x(t_b) = x_b$$

we can "Taylor" expand just as we did in the calculus of variations, $x(t) = x_0(t) + \delta x(t)$, where $x_0(t)$ is the minimising path & $\delta x(t)$ parameterises deviations from this $\delta x(t_a) = 0$, $\delta x(t_b) = 0$

$$S = S[x_0] + \int_{t_a}^{t_b} dt \left.\frac{\partial L}{\partial \dot{x}}\right|_{x_0} \delta \dot{x} + \frac{1}{2}\int_{t_a}^{t_b} dt \left.\frac{\partial^2 L}{\partial \dot{x}^2}\right|_{x_0} (\delta \dot{x})^2 + 0$$

since $\dfrac{\partial^3 L}{\partial \dot{x}^3} = 0$ as are higher powers.

the minimising path is the classical trajectory, so by the Euler-Lagrange equations $\dfrac{\partial L}{\partial \dot{x}} = 0$

$$S = S[x_0] + \frac{1}{2}\int_{t_a}^{t_b} dt \left.\frac{\partial^2 L}{\partial \dot{x}^2}\right|_{x_0} (\delta \dot{x})^2 \qquad x_0(t) = \frac{x_b(t-t_a) - x_a(t-t_b)}{t_b - t_a}$$

$$L\big|_{x_0} = \frac{1}{2} m \dot{x}_0^2 = \frac{1}{2} m \left(\frac{x_b - x_a}{t_b - t_a}\right)^2$$

$$S[x_0] = \frac{1}{2} m \frac{(x_b - x_a)^2}{t_b - t_a}$$

$$K(x_b t_b ; x_a t_a) = e^{iS[x_0]/\hbar} \cdot \int \mathcal{D}[\delta x(t)] \exp\left[\frac{i}{2\hbar}\int_{t_a}^{t_b} dt \frac{\partial^2 L}{\partial \dot{x}^2} (\delta \dot{x})^2\right]$$

notice that $\dfrac{\partial^2 L}{\partial \dot{x}^2} = m$

so the 2nd factor is $\int \mathcal{D}[\delta x(t)] \exp\left[\frac{i}{\hbar}\int_{t_a}^{t_b} dt \ \frac{1}{2} m (\delta \dot{x})^2\right]$

hence the propagator for propagation from 0 @ t_a to 0 @ t_b

$$\boxed{K(x_b t_b ; x_a t_a) = e^{iS[x_0]/\hbar} \cdot K(0 t_b ; 0 t_a) = K(0, t_b ; 0 t_a) \, e^{i \frac{m(x_b - x_a)^2}{2\hbar(t_b - t_a)}}}$$

we can fully determine this by explicitly computing the path integral using the discrete form given earlier.

$$I_{N,\varepsilon} = \int dx_1 dx_2 \cdots dx_N \ \exp\left(\frac{i\varepsilon}{\hbar} \sum_{n=1}^{N+1} \frac{1}{2} m \dot{x}_n\right)$$

$$= \int dx_1 dx_2 \cdots dx_N \ \exp\left(\frac{i\varepsilon}{\hbar} \frac{m}{2}\left(\frac{(x_b - x_N)^2}{\varepsilon^2} + \frac{(x_N - x_{N-1})^2}{\varepsilon^2} + \cdots \frac{(x_1 - x_a)^2}{\varepsilon^2}\right)\right)$$

$$I^{(x_1)} = \int dx_1 \; \exp\left(\frac{im}{2\hbar\varepsilon}\left((x_2 - x_1)^2 + (x_1 - x_a)^2\right)\right) = \int dx_1 \; \exp\left(\frac{im}{2\hbar\varepsilon}\left[x_2^2 - 2x_2 x_1 + 2x_1^2 - 2x_1 x_a + x_a^2\right]\right)$$

$$= \int dx_1 \exp\left(\frac{im}{2\varepsilon\hbar}\left[x_2^2 + x_a^2 + 2\left(x_1^2 - x_1(x_2 + x_a)\right)\right]\right)$$

$$x_1^2 - x_1(x_2 + x_a) = \left(x_1 - \tfrac{1}{2}(x_2 + x_a)\right)^2 - \tfrac{1}{4}(x_2 + x_a)^2$$

$$= \exp\left(\frac{im}{2\varepsilon\hbar}\left(x_2^2 + x_a^2 - \frac{1}{2}(x_2 + x_a)^2\right)\right) \underbrace{\int dx_1 \; \exp\left(\frac{im}{\varepsilon\hbar}\left(x_1 - \tfrac{1}{2}(x_2 + x_a)\right)^2\right)}_{\sqrt{\pi}\Big/\sqrt{\frac{-im}{\varepsilon\hbar}}}$$

$$I^{(x_1)} = \sqrt{\frac{i\pi\varepsilon\hbar}{m}} \; \exp\left(\frac{im}{4\varepsilon\hbar}(x_2 - x_a)^2\right)$$

multiply by the remaining x_2 dependence & integrate over dx_2

$$\sqrt{\frac{i\pi\varepsilon\hbar}{m}} \int dx_2 \cdot \exp\left(\frac{im}{4\varepsilon\hbar}(x_2 - x_a)^2\right) \cdot \exp\left(\frac{im}{2\varepsilon\hbar}(x_3 - x_2)^2\right)$$

$$= \sqrt{\frac{i\pi\varepsilon\hbar}{m}} \int dx_2 \; \exp\left(\frac{im}{4\varepsilon\hbar}\left[x_2^2 - 2x_2 x_a + x_a^2 + 2\left(x_3^2 - 2x_3 x_2 + x_2^2\right)\right]\right)$$

$$= \sqrt{\frac{i\pi\varepsilon\hbar}{m}} \exp\left(\frac{im}{4\varepsilon\hbar}\left[x_a^2 + 2x_3^2\right]\right) \int dx_2 \; \exp\left(\frac{im}{4\varepsilon\hbar}\left[3x_2^2 - 2x_2 x_a - 4x_3 x_2\right]\right)$$

$$3x_2^2 - 2x_2(x_a + 2x_3)$$
$$= 3\left(x_2^2 - \tfrac{2}{3}x_2(x_a + 2x_3)\right)$$
$$= 3\left(\left[x_2 - \tfrac{1}{3}(x_a + 2x_3)\right]^2 - \tfrac{1}{9}(x_a + 2x_3)^2\right)$$

$$= \sqrt{\frac{i\pi\varepsilon\hbar}{m}} \; \exp\left(\frac{im}{4\varepsilon\hbar}\left[x_a^2 + 2x_3^2 - \tfrac{1}{3}(x_a + 2x_3)^2\right]\right) \underbrace{\int dx_2 \; \exp\left(\frac{3im}{4\varepsilon\hbar}\left(x_2 - \tfrac{1}{3}(x_a + 2x_3)\right)^2\right)}_{\sqrt{\pi}\Big/\sqrt{\frac{-3im}{4\varepsilon\hbar}} = \sqrt{\frac{4i\pi\varepsilon\hbar}{3m}}}$$

$$= \sqrt{\frac{i\pi\varepsilon\hbar}{m}}\sqrt{\frac{4i\pi\varepsilon\hbar}{3m}} \; \exp\left(\frac{im}{4\varepsilon\hbar}\frac{2}{3}(x_3 - x_a)^2\right) = \sqrt{\frac{i\pi\varepsilon\hbar}{m}}\sqrt{\frac{4i\pi\varepsilon\hbar}{3m}} \; \exp\left(\frac{im}{6\varepsilon\hbar}(x_3 - x_a)^2\right)$$

by induction $I_{N,\varepsilon} = \dfrac{1}{\sqrt{N+1}}\left(\dfrac{2\pi i\varepsilon\hbar}{m}\right)^{N/2} \exp\left(\dfrac{im}{2\varepsilon\hbar(N+1)}(x_b - x_a)^2\right)$

$$K(x_b t_b ; x_a t_a) = \lim_{\substack{N\to\infty \\ \varepsilon\to 0}} \left(\frac{m}{2\pi i \varepsilon \hbar}\right)^{\frac{N+1}{2}} I_{N,\varepsilon} = \lim_{\substack{N\to\infty \\ \varepsilon\to 0}} \sqrt{\frac{m}{2\pi i \hbar \varepsilon (N+1)}} \exp\left(\frac{im}{2\hbar \varepsilon (N+1)} (x_b - x_a)^2\right)$$

$$= \underline{\sqrt{\frac{m}{2\pi i \hbar (t_b - t_a)}} \exp\left[\frac{im}{2\hbar} \frac{(x_b - x_a)^2}{t_b - t_a}\right]}$$

Harmonic oscillator in the path integral formalism

The Lagrangian for the harmonic oscillator in one-dimension is

$$L = \tfrac{1}{2}m\dot{x}^2 - \tfrac{1}{2}m\omega^2 x^2$$

expanding the action about the classical solution, $x_0(t)$ we have

$$S[x] = \int_{t_a}^{t_b} dt\, L = S[x_0] + \frac{1}{2}\int dt\, \frac{\partial^2 L}{\partial x^2}[\delta x(t)]^2 + \frac{1}{2}\int dt\, \frac{\partial^2 L}{\partial \dot{x}^2}[\delta \dot{x}(t)]^2$$

$$\left(x(t) = x_0(t) + \delta x(t)\right) \qquad = S[x_0] + \frac{m}{2}\int dt\left([\delta\dot{x}(t)]^2 - \omega^2[\delta x(t)]^2\right)$$

Suppose $t_a = 0$, $t_b = T$
$x(0) = x_a$, $x(T) = x_b$
$\delta x(0) = 0$, $\delta x(T) = 0$

$$K(x_b, T; x_a 0) = e^{iS[x_0]/\hbar}\int \mathcal{D}(\delta x)\exp\left(\frac{im}{2\hbar}\int_0^T dt\left(\delta\dot{x}^2 - \omega^2\delta x^2\right)\right)$$

$$= e^{iS[x_0]/\hbar}\cdot K(0,T;0,0)$$

since the second term is the path integral over all paths starting at 0 @ 0 and ending at 0 @ T.

So what is the <u>classical action</u> for the harmonic oscillator with the above boundary conditions?

The classical trajectory is $x(t) = A\sin(\omega t + \phi)$ with $x_a = A\sin\phi$ & $x_b = A\sin(\omega T + \phi)$ determining A & ϕ.

$$S[x_0] = \frac{m}{2}\int_0^T dt\left(\dot{x}^2 - \omega^2 x^2\right) = \frac{m}{2}\int_0^T dt\left(\omega^2A^2\cos^2(\omega t+\phi) - \omega^2A^2\sin^2(\omega t+\phi)\right)$$

$$= \frac{m\omega^2}{2}A^2\int_0^T dt\cos(2\omega t + 2\phi) = \frac{m\omega^2}{2}A^2\cdot\frac{1}{2\omega}\left(\sin(2\omega T + 2\phi) - \sin 2\phi\right)$$

$$= \frac{1}{4}m\omega A^2\left(\sin(2\omega T+2\phi) - \sin 2\phi\right) \xrightarrow[\text{algebra}]{\text{some}} \underline{\frac{m\omega}{2\sin\omega T}\left((x_a^2 + x_b^2)\cos\omega T - 2x_a x_b\right)}$$

to determine the normalisation of the propagator we need to know $K(0,T;0,0)$

$$K(0,T;0,0) = \int \mathcal{D}y\,\exp\left[\frac{i}{\hbar}\int_0^T dt\left(\tfrac{1}{2}m\dot{y}^2 - \tfrac{1}{2}m\omega^2y^2\right)\right] \equiv \lim_{\substack{N\to\infty \\ \varepsilon\to 0 \\ N\varepsilon\text{ const}}}\left\{\left(\frac{m}{2\pi\hbar i\varepsilon}\right)^{\frac{N+1}{2}} I_{N,\varepsilon}\right\}$$

$$\begin{bmatrix} y(0) = 0 \\ y(T) = 0 \end{bmatrix}$$

where $I_{N,\varepsilon} = \int dy_1 dy_2 \dots dy_N \exp\left(\frac{i\varepsilon}{\hbar} \frac{m}{2} \sum_{n=1}^{N} (\dot{y}_n^2 - \omega^2 y_n^2) \right)$

$$\varepsilon \sum_n (\dot{y}_n^2 - \omega^2 y_n^2) \to \int dt\,(\dot{y}^2 - \omega^2 y^2) = \int dt\,(-y\ddot{y} - \omega^2 y) \quad \text{by an integration by parts}$$

$$= -\int dt\, y \underbrace{\left[\frac{d^2}{dt^2} + \omega^2\right]}_{M} y$$

using finite differences to represent the operator $\frac{d}{dt}$, I can turn this back into a matrix problem

$$\to -\varepsilon \sum_n \sum_k y_k M_{kn} y_n$$

M is a real, symmetric matrix, so I can diagonalise it if I know the eigenvectors. $MX = \lambda X$.
The eigenvectors form an orthogonal matrix, so the transformation from the basis y_n to x_n is just a "rotation" and has unit Jacobean

$$\Rightarrow I \to \int dx_1 dx_2 \dots dx_N \exp\left(-\frac{i\varepsilon}{\hbar} \frac{m}{2} \sum_n \lambda_n x_n^2 \right)$$

$$= \prod_{n=1}^{N} \left(\frac{\pi}{\frac{i\varepsilon m}{2\hbar} \lambda_n} \right)^{1/2} = \left(\frac{2\pi\hbar}{im\varepsilon} \right)^{N/2} \cdot \frac{1}{\sqrt{\det M}}$$

det M ?

$$M = \frac{d^2}{dt^2} + \omega^2 \quad \Rightarrow \text{eigenfunctions satisfy}$$

$$\left(\frac{d^2}{dt^2} + \omega^2\right) x_n(t) = \lambda_n x_n(t)$$

so $x_n(t) = A \sin(\omega_n t + \phi)$ & $\lambda_n = \omega^2 - \omega_n^2$

but $x_n(0) = 0 = A\sin\phi \Rightarrow \phi = 0$

$x_n(T) = 0 = A \sin \omega_n T \Rightarrow \omega_n T = n\pi$

$\underline{\omega_n = n\pi/T}$

$\underline{\lambda_n = \omega^2 - \left(\frac{n\pi}{T}\right)^2}$

$$\det M = \prod_{n=1}^{N\to\infty} \lambda_n = \prod_{n=1}^{N\to\infty} \left(\omega^2 - \left(\frac{n\pi}{T}\right)^2\right) = \prod_{n=1}^{\infty} -\left(\frac{n\pi}{T}\right)^2 \left(1 - \left(\frac{\omega T}{n\pi}\right)^2\right)$$

$$= \underbrace{(-1)^N \left(\frac{\pi}{T}\right)^2 \left(\frac{2\pi}{T}\right)^2 \left(\frac{3\pi}{T}\right)^2 \cdots \frac{1}{\omega T}}_{\text{constant, } C} \underbrace{\omega T \left(1 - \left(\frac{\omega T}{\pi}\right)^2\right)\left(1 - \left(\frac{\omega T}{2\pi}\right)^2\right)\left(1 - \left(\frac{\omega T}{3\pi}\right)^2\right) \cdots}_{F(\omega T)\ ?}$$

independent of ω

notice that $F(\omega T)$ has zeroes when $\omega T = n\pi$ & has no poles – this looks just like the function $\sin\omega T$. A much more careful analysis shows that this is the case:

$$\det M = C \cdot \frac{\sin\omega T}{\omega T} \quad \Rightarrow \quad \frac{1}{\sqrt{\det M}} = \frac{1}{\sqrt{C}}\sqrt{\frac{\omega T}{\sin\omega T}}$$

collecting everything independent of ω into an overall normalisation we have

$$K(0,T;0,0) = C'\sqrt{\frac{\omega T}{\sin\omega T}}$$

$$\& \quad K(x_b T; x_a 0) = C'\sqrt{\frac{\omega T}{\sin\omega T}}\exp\left(\frac{im\omega}{2\hbar\sin\omega T}\left((x_a^2+x_b^2)\cos\omega T - 2x_a x_b\right)\right)$$

we can determine the norm C' by insisting that as $\omega\to 0$, we tend to the free particle propagator

$$K(x_b T, x_a 0) \xrightarrow[\omega\to 0]{} C' \cdot 1 \cdot \exp\left(\frac{im}{2\hbar T}\cdot 1 \cdot (x_a - x_b)^2\right)$$

$$\Rightarrow C' = \sqrt{\frac{m}{2\pi i\hbar T}}$$

$$\& \quad \boxed{K_{HO}(x_b T; x_a 0) = \sqrt{\frac{m\omega}{2\pi i\hbar\sin\omega T}}\exp\left(\frac{im\omega}{2\hbar\sin\omega T}\left[(x_a^2+x_b^2)\cos\omega T - 2x_a x_b\right]\right)}$$

the propagator contains a lot of useful information - one nice way to get at that information is to perform an analytic continuation from real T to imaginary T, $T = -i\tau$ "Euclidean time"

then $K(x_b T; x_a 0) = \langle x_b | e^{-iHT/\hbar} | x_a \rangle \rightarrow \langle x_b | e^{-H\tau/\hbar} | x_a \rangle = \tilde{K}(x_b \tau; x_a 0)$

e.g. inserting the complete set of eigenstates of H $\quad (H|n\rangle = E_n |n\rangle)$

$$\tilde{K}(x_b \tau; x_a 0) = \sum_n \langle x_b | e^{-iH\tau/\hbar} | n\rangle\langle n | x_a \rangle = \sum_n \psi_n^*(x_b)\, \psi_n(x_a)\, e^{-E_n \tau/\hbar}$$

$$\xrightarrow[\tau\to\infty]{} \psi_0^*(x_b)\, \psi_0(x_a)\, e^{-E_0 \tau/\hbar}$$ where $n=0$ is the g-s.

so just by knowing the large "Euclidean time" behaviour of the propagator we can extract the ground state wavefunction & energy

e.g. the HO:

$$\tilde{K}(x_b \tau; x_a 0) = \sqrt{\frac{m\omega}{2\pi\hbar \sinh\omega\tau}}\, \exp\left(-\frac{m\omega}{2\hbar \sinh\omega\tau} \left[(x_a^2 + x_b^2)\cosh\omega\tau - 2x_a x_b \right] \right) \quad (A)$$

as $\tau \to \infty$ $\quad \sinh\omega\tau \to \frac{1}{2} e^{\omega\tau}$ $\quad$ & $\cosh\omega\tau \to \frac{1}{2} e^{\omega T}$

$$\Rightarrow \lim_{\tau\to\infty} \tilde{K}(x_b \tau; x_a 0) = \sqrt{\frac{m\omega}{\pi\hbar}}\, e^{-\omega\tau/2} \cdot \exp\left(-\frac{m\omega}{2\hbar}\, 2e^{-\omega\tau} \left[\tfrac{1}{2}(x_a^2 + x_b^2) e^{\omega\tau} - 2x_a x_b \right] \right)$$

$$= \left[\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar} x_a^2} \right] \left[\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar} x_b^2} \right] e^{-(\frac{1}{2}\hbar\omega)\cdot \tau/\hbar}$$

$$\Rightarrow \underline{\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar} x^2}} \quad \& \quad \underline{E_0 = \tfrac{1}{2}\hbar\omega}$$

In fact the τ-dependence of the propagator can give us the entire spectrum

expanding the exponent of (A) $\quad -\frac{m\omega}{2\hbar} \frac{2}{e^{\omega\tau} - e^{-\omega\tau}} \left[(x_a^2 + x_b)^2 \tfrac{1}{2}(e^{\omega\tau} + e^{-\omega\tau}) - 2x_a x_b \right]$

$$= -\frac{m\omega}{\hbar} \frac{1}{\cancel{e^{\omega\tau}}} \frac{1}{1 - e^{-2\omega\tau}} \frac{1}{2} \cancel{e^{\omega\tau}} \left[(x_a^2 + x_b^2)(1 + e^{-2\omega\tau}) - 4x_a x_b e^{-\omega\tau} \right]$$

$\xi = e^{-\omega\tau}$

$$= -\frac{m\omega}{2\hbar} (1 + \xi^2 + \dots) \left[(x_a^2 + x_b^2)(1 + \xi^2) - 4x_a x_b \xi \right]$$

to lowest non-trivial order in ξ : $-\frac{m\omega}{2\hbar}(x_a^2+x_b^2) + \frac{2m\omega}{\hbar}x_a x_b \xi + \mathcal{O}(\xi^2)$

$$\Rightarrow \quad \tilde{K}(x_b\,\tau; x_a\,0) \longrightarrow \sqrt{\frac{m\omega}{\pi\hbar}}\, e^{-\omega\tau/2}\, e^{-\frac{m\omega}{2\hbar}(x_a^2+x_b^2)}\, e^{\frac{2m\omega}{\hbar}x_a x_b \xi}$$

$$\longrightarrow e^{-\omega\tau/2}\sum_n A_n \xi^n = e^{-\omega\tau/2}\sum_n A_n \left(e^{-\omega\tau}\right)^n$$

$$\sim e^{-(n+\frac{1}{2})\hbar\omega\cdot\tau/\hbar} \qquad \Rightarrow \underline{E_n = \hbar\omega(n+\tfrac{1}{2})}$$