ADDITION OF ANGULAR MOMENTUM

Recall that a spin-½ particle state can be described by eigenkets of operators $\hat{S}^2$ & $\hat{S}_z$:

$$\hat{S}^2 |\tfrac{1}{2}, \pm\tfrac{1}{2}\rangle = \hbar^2 \tfrac{1}{2}(\tfrac{1}{2}+1)|\tfrac{1}{2}, \pm\tfrac{1}{2}\rangle = \tfrac{3}{4}\hbar^2 |\tfrac{1}{2}, \pm\tfrac{1}{2}\rangle$$

$$\hat{S}_z |\tfrac{1}{2}, \pm\tfrac{1}{2}\rangle = \pm\tfrac{1}{2}\hbar |\tfrac{1}{2}, \pm\tfrac{1}{2}\rangle.$$

Under a rotation, states transform by multiplication of the rotation operator:

$$|\alpha\rangle_R = \mathcal{D}(R)|\alpha\rangle \qquad \text{where} \qquad \mathcal{D}(R) = \exp\left[-i \frac{\hat{n}\cdot\hat{\vec{S}}}{\hbar}\phi\right]$$ for a rotation of ϕ about the axis $\hat{n}$.

Now we will consider a system of <u>two</u> spin-½ particles. The state of this two-particle system can be described by the direct product state of the two single-particle states:

$$\left|\tfrac{1}{2}, \pm\tfrac{1}{2}\right\rangle_{(1)} \otimes \left|\tfrac{1}{2}, \pm\tfrac{1}{2}\right\rangle_{(2)} \equiv \left|\tfrac{1}{2}, \pm\tfrac{1}{2}; \tfrac{1}{2}, \pm\tfrac{1}{2}\right\rangle$$

Clearly this state is still an eigenstate of the operators $\hat{S}^2_{(1)}, \hat{S}_{(1)z}, \hat{S}^2_{(2)}, \hat{S}_{(2)z}$, where e.g. $\hat{S}_{(1)z}$ acts only on the (1) part of the state.

The transformation of the direct product state under a rotation requires use of the direct product rotation operator:

$$\mathcal{D}_{(1)}(R) \otimes \mathcal{D}_{(2)}(R) = \exp\left[-i\frac{\hat{n}\cdot\hat{\vec{S}}_{(1)}}{\hbar}\phi\right] \otimes \exp\left[-i\frac{\hat{n}\cdot\hat{\vec{S}}_{(2)}}{\hbar}\phi\right].$$

We can define a <u>total spin operator</u> $\hat{\vec{S}} = \hat{\vec{S}}_{(1)} + \hat{\vec{S}}_{(2)}$,

where really this means $\hat{\vec{S}} = \hat{\vec{S}}_{(1)} \otimes \hat{1}_{(2)} + \hat{1}_{(1)} \otimes \hat{\vec{S}}_{(2)}$

acts on the state of particle (2).

acts on the state of particle (1).

The spin operators for particles (1) & (2) each satisfy the appropriate commutation relations

$$[\hat{S}_{(1)i}, \hat{S}_{(1)j}] = i\epsilon_{ijk}\hbar\hat{S}_{(1)k} \quad \& \quad [\hat{S}_{(2)i}, \hat{S}_{(2)j}] = i\epsilon_{ijk}\hbar\hat{S}_{(2)k}$$

& the operators for different particles commute:

$$[\hat{S}_{(1)i}, \hat{S}_{(2)j}] = 0$$

As a consequence $[\hat{S}_i, \hat{S}_j] = i\epsilon_{ijk}\hbar\hat{S}_k$, so the total "spin" operator also satisfies the commutation relation for angular momentum.

Since $\hat{\vec{S}}$ satisfies angular momentum commutation relations we already know its eigenstates:

$$\hat{\vec{S}}^2 |s,m\rangle = s(s+1)\hbar^2 |s,m\rangle$$
$$\hat{S}_z |s,m\rangle = m\hbar |s,m\rangle .$$

Thus we have two bases in which we can consider the two particle states:
- the direct product of $\hat{\vec{S}}_{(1)}, \hat{\vec{S}}_{(2)}$ eigenstates;
- the eigenstates of the total spin operator $\hat{\vec{S}}$.

e.g. consider the direct product state $|\frac{1}{2}, +\frac{1}{2}; \frac{1}{2}, +\frac{1}{2}\rangle \equiv |++\rangle$

this is also an eigenstate of $\hat{S}^2$ & $\hat{S}_z$ – let's prove this:

$$\hat{S}^2 = \left(\hat{\vec{S}}_{(1)} + \hat{\vec{S}}_{(2)}\right)^2 = \hat{S}_{(1)}^2 + \hat{S}_{(2)}^2 + 2\,\hat{\vec{S}}_{(1)}\cdot\hat{\vec{S}}_{(2)}$$

$$= \hat{S}_{(1)}^2 + \hat{S}_{(2)}^2 + 2\left(\hat{S}_{(1)x}\hat{S}_{(2)x} + \hat{S}_{(1)y}\hat{S}_{(2)y} + \hat{S}_{(1)z}\hat{S}_{(2)z}\right)$$

$$\hat{S}^2 = \hat{S}_{(1)}^2 + \hat{S}_{(2)}^2 + 2\hat{S}_{(1)z}\hat{S}_{(2)z} + \hat{S}_{(1)+}\hat{S}_{(2)-} + \hat{S}_{(1)-}\hat{S}_{(2)+}$$

so $\hat{S}^2|++\rangle = \frac{1}{2}(\frac{1}{2}+1)\hbar^2|++\rangle + \frac{1}{2}(\frac{1}{2}+1)\hbar^2|++\rangle + 2\cdot(+\frac{1}{2}\hbar)(+\frac{1}{2}\hbar)|++\rangle + 0 + 0$

$= 2\hbar^2|++\rangle = 1(1+1)\hbar^2|++\rangle$ $\Rightarrow$ eigenstate with $s=1$

& $\hat{S}_z|++\rangle = \left(\hat{S}_{(1)z} + \hat{S}_{(2)z}\right)|++\rangle = \left(\hbar(+\frac{1}{2}) + \hbar(+\frac{1}{2})\right)|++\rangle = \hbar|++\rangle$

$\Rightarrow$ eigenstate with $m=+1$

$$\Rightarrow \underline{|\tfrac{1}{2}, +\tfrac{1}{2}; \tfrac{1}{2}, +\tfrac{1}{2}\rangle = |s=1, m=+1\rangle}$$

This appears to be an eigenstate with spin 1. We should be able to construct a state $|s=1, m=0\rangle$ by acting with $\hat{S}_-$:

$$\hat{S}_-|s=1; m=+1\rangle = \hbar\sqrt{1\cdot(1+1) - 1(1-1)}\,|s=1, m=0\rangle = \sqrt{2}\,\hbar\,|s=1, m=0\rangle$$

$$|s=1; m=0\rangle = \frac{1}{\sqrt{2}}\frac{1}{\hbar}\hat{S}_-|s=1, m=+1\rangle = \frac{1}{\hbar\sqrt{2}}\left(\hat{S}_{(1)-} + \hat{S}_{(2)-}\right)|++\rangle = \frac{1}{\hbar\sqrt{2}}\left(\hbar|-+\rangle + \hbar|+-\rangle\right)$$

$$\Rightarrow \underline{|s=1; m=0\rangle = \frac{1}{\sqrt{2}}\left(|\tfrac{1}{2}, +\tfrac{1}{2}; \tfrac{1}{2}, -\tfrac{1}{2}\rangle + |\tfrac{1}{2}, -\tfrac{1}{2}; \tfrac{1}{2}, +\tfrac{1}{2}\rangle\right)}$$

notice that this is NOT an eigenstate of $\hat{S}_{(1)z}$ or $\hat{S}_{(2)z}$,
but it is a simultaneous eigenstate of $\hat{S}^2, \hat{S}_z, \hat{S}_{(1)}^2, \hat{S}_{(2)}^2$

It is easy to see that this is because $[\hat{S}^2, \hat{S}_z] = [\hat{S}^2, \hat{S}_{(1)}^2] = [\hat{S}^2, \hat{S}_{(2)}^2]$

$$= [\hat{S}_z, \hat{S}_{(1)}^2] = [\hat{S}_z, \hat{S}_{(2)}^2] = \underline{0}$$

but $[\hat{S}^2, \hat{S}_{(1)z}] \neq 0$

$[\hat{S}^2, \hat{S}_{(2)z}] \neq 0$

We can easily convince ourselves that $\underline{|s=1, m=1\rangle = \left|\tfrac{1}{2}, -\tfrac{1}{2}; \tfrac{1}{2}, -\tfrac{1}{2}\right\rangle} = |--\rangle$

But notice that while we have four states in the $(1)\otimes(2)$ basis: $|++\rangle, |+-\rangle, |-+\rangle, |--\rangle$, we appear to have only three in the $\hat{S}^2, \hat{S}_z$ basis:

$$|++\rangle = |1,+1\rangle$$

$$\frac{1}{\sqrt{2}}(|+-\rangle + |-+\rangle) = |1,0\rangle \qquad ?$$

$$|--\rangle = |1,-1\rangle$$

We can construct a fourth state orthogonal to the previous three; it is, fairly obviously $|\psi\rangle = \frac{1}{\sqrt{2}}(|+-\rangle - |-+\rangle)$. What are its properties under the action of $\hat{S}^2$ & $\hat{S}_z$?

$$\hat{S}^2|\psi\rangle = \frac{3}{4}\hbar^2|\psi\rangle + \frac{3}{4}\hbar^2|\psi\rangle + 2\frac{1}{\sqrt{2}}\left(\left(\frac{\hbar}{2}\right)\left(-\frac{\hbar}{2}\right)|+-\rangle - \left(-\frac{\hbar}{2}\right)\left(\frac{\hbar}{2}\right)|-+\rangle\right)$$

$$+ \frac{1}{\sqrt{2}}\left(0 - \hbar\hbar|+-\rangle\right) + \frac{1}{\sqrt{2}}\left(\hbar\hbar|-+\rangle + 0\right)$$

$$= \hbar^2\left(\frac{3}{2} - \frac{1}{2} - 1\right)|\psi\rangle = 0 \cdot |\psi\rangle \qquad \Rightarrow \underline{\text{eigenstate with } s=0}$$

$$\hat{S}_z|\psi\rangle = \left(\hat{S}_{(1)z} + \hat{S}_{(2)z}\right)|\psi\rangle = \frac{1}{\sqrt{2}}\left(\cancel{\frac{\hbar}{2}|+-\rangle} - \cancel{\left(\frac{\hbar}{2}\right)|-+\rangle}\right) + \frac{1}{\sqrt{2}}\left(\cancel{-\frac{\hbar}{2}|+-\rangle} - \cancel{\frac{\hbar}{2}|-+\rangle}\right) = 0|\psi\rangle$$

$\Rightarrow$ <u>eigenstate with $m=0$</u>

$$\Rightarrow \quad \underline{|s=0, m=0\rangle = \frac{1}{\sqrt{2}}\left(\left|\tfrac{1}{2}, +\tfrac{1}{2}; \tfrac{1}{2}, -\tfrac{1}{2}\right\rangle - \left|\tfrac{1}{2}, -\tfrac{1}{2}; \tfrac{1}{2}, +\tfrac{1}{2}\right\rangle\right)}$$

Hence two spin-½ particles can be coupled to give a system which overall has spin-1 or spin-0. The transformation from the $(1)\otimes(2)$ basis to the $|s,m\rangle$ basis is effected by a <u>unitary matrix</u> whose entries are known as the 'Clebsch-Gordan coefficients':

$$\begin{bmatrix} |s=0, m=0\rangle \\ |s=1, m=+1\rangle \\ |s=1, m=0\rangle \\ |s=1, m=-1\rangle \end{bmatrix} = \begin{bmatrix} 0 & 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} |++\rangle \\ |+-\rangle \\ |-+\rangle \\ |--\rangle \end{bmatrix}$$

* Addition of angular momentum - the general case:

Consider two angular momentum operators which act in different subspaces:

$\hat{\vec{J}}_{(1)}$, $\hat{\vec{J}}_{(2)}$ where $[\hat{J}_{(1)i}, \hat{J}_{(1)j}] = i\epsilon_{ijk}\hbar \hat{J}_{(1)k}$ (A)

$[\hat{J}_{(2)i}, \hat{J}_{(2)j}] = i\epsilon_{ijk}\hbar \hat{J}_{(2)k}$ (B)

& $[\hat{J}_{(1)i}, \hat{J}_{(2)j}] = 0$ (C)

The infinitesimal rotation operator for the product space is

$$\left(\hat{1}_{(1)} - \tfrac{i}{\hbar}\hat{n}\cdot\hat{\vec{J}}_{(1)}\delta\phi\right) \otimes \left(\hat{1}_{(2)} - \tfrac{i}{\hbar}\hat{n}\cdot\hat{\vec{J}}_{(2)}\delta\phi\right)$$

$$= \hat{1}_{(1)} \otimes \hat{1}_{(2)} - \frac{i}{\hbar}\hat{n}\cdot\left(\hat{\vec{J}}_{(1)}\otimes\hat{1}_{(2)} + \hat{1}_{(1)}\otimes\hat{\vec{J}}_{(2)}\right)\delta\phi$$

which has the form $\hat{1} - \tfrac{i}{\hbar}\hat{n}\cdot\hat{\vec{J}}\,\delta\phi$ where $\hat{\vec{J}} \equiv \hat{\vec{J}}_{(1)}\otimes\hat{1}_{(2)} + \hat{1}_{(1)}\otimes\hat{\vec{J}}_{2}$

or $\hat{\vec{J}} = \hat{\vec{J}}_{(1)} + \hat{\vec{J}}_{(2)}$ as shorthand

we call this the <u>total</u> angular momentum operator.
From its appearance as the generator of rotations we'd assume it satisfies the angular momentum commutation relations. To be certain we can check it does using (A), (B), (C)

$$\Rightarrow [\hat{J}_i, \hat{J}_j] = i\epsilon_{ijk}\hbar\hat{J}_k$$

There are clearly two possible basis sets:

<u>PRODUCT BASIS</u>:	<u>TOTAL $\vec{J}$ BASIS</u>:
eigenstates of $\hat{J}^2_{(1)}, \hat{J}_{(1)z}$ & $\hat{J}^2_{(2)}, \hat{J}_{(2)z}$	eigenstates of $\hat{J}^2, \hat{J}_z$. note that since $[\hat{J}^2, \hat{J}^2_{(1)}] = [\hat{J}^2, \hat{J}^2_{(2)}] = 0$ they can also be eigenstates of $\hat{J}^2_{(1)}, \hat{J}^2_{(2)}$
$\lvert j_1 m_1; j_2 m_2\rangle \equiv \lvert j_1 m_1\rangle \otimes \lvert j_2 m_2\rangle$	$\lvert [j_1 j_2]; jm\rangle$
$\hat{J}^2_{(1)}\lvert j_1 m_1; j_2 m_2\rangle = j_1(j_1+1)\hbar^2\lvert j_1 m_1; j_2 m_2\rangle$	$\hat{J}^2_{(1)}\lvert [j_1 j_2]; jm\rangle = j_1(j_1+1)\hbar^2\lvert [j_1 j_2]; jm\rangle$
$\hat{J}_{(1)z}\lvert j_1 m_1; j_2 m_2\rangle = m_1\hbar\lvert j_1 m_1; j_2 m_2\rangle$	$\hat{J}^2_{(2)}\lvert [j_1 j_2]; jm\rangle = j_2(j_2+1)\hbar^2\lvert [j_1 j_2]; jm\rangle$
$\hat{J}^2_{(2)}\lvert j_1 m_1; j_2 m_2\rangle = j_2(j_2+1)\hbar^2\lvert j_1 m_1; j_2 m_2\rangle$	$\hat{J}^2\lvert [j_1 j_2]; jm\rangle = j(j+1)\hbar^2\lvert [j_1 j_2]; jm\rangle$
$\hat{J}_{(2)z}\lvert j_1 m_1; j_2 m_2\rangle = m_2\hbar\lvert j_1 m_1; j_2 m_2\rangle$	$\hat{J}_z\lvert [j_1 j_2]; jm\rangle = m\hbar\lvert [j_1 j_2]; jm\rangle$

Each of these bases is a complete set of states so they should be connected by a unitary transformation,

e.g. $\sum_{m_1 m_2} |j_1 m_1; j_2 m_2\rangle\langle j_1 m_1; j_2 m_2| = \mathbb{1}$ (for a fixed $j_1 j_2$ pair)

so $|[j_1 j_2]; jm\rangle = \sum_{m_1 m_2} \underbrace{\langle j_1 m_1, j_2 m_2 | [j_1 j_2]; jm\rangle}_{\text{"Clebsch-Gordan coefficients"}} \cdot |j_1 m_1; j_2 m_2\rangle$

Properties of the Clebsch-Gordan coefficients:

* they are zero unless $m = m_1 + m_2$

$\hookrightarrow \left(\hat{J}_z - \hat{J}_{(1)z} - \hat{J}_{(2)z}\right)|[j_1 j_2]; jm\rangle = 0$

$\Rightarrow \langle j_1 m_1; j_2 m_2 | \left(\hat{J}_z - \hat{J}_{(1)z} - \hat{J}_{(2)z}\right)|[j_1 j_2]; jm\rangle = 0$

$\Rightarrow \hbar(m - m_1 - m_2)\langle j_1 m_1; j_2 m_2 | [j_1 j_2]; jm\rangle = 0$

$\Rightarrow \langle j_1 m_1; j_2 m_2 | [j_1 j_2]; jm\rangle \propto \delta_{m, m_1 + m_2}$

* the CGC vanish unless $|j_1 - j_2| \leq j \leq |j_1 + j_2|$

$\hookrightarrow$ complete proof is long — let's just check this gives the right dimensionality of the state space

→ PRODUCT SPACE: $|j_1 m_1; j_2 m_2\rangle$ $\quad N = (2j_1+1)(2j_2+1)$

→ TOTAL J SPACE: $|[j_1 j_2]; jm\rangle$ $(j_2 \geq j_1)$ $\quad N = \sum_{j = j_2 - j_1}^{j_2 + j_1} (2j+1)$ $\quad \left[\sum_{n=0}^{m} n = \frac{1}{2}m(m+1)\right]$

$= \sum_{k=0}^{2j_1} 2[k + (j_2 - j_1)] + 1 = 2\sum_{k=0}^{2j_1} k + \left(2(j_2 - j_1) + 1\right)\sum_{k=0}^{2j_1} 1$

$= 2j_1(2j_1+1) + \left(2(j_2 - j_1) + 1\right)\cdot(2j_1+1)$

$= (2j_1+1)(2j_2+1)$ ✓

* with the phase convention of Edmonds that all CGCs are real, the CGCs form an orthogonal matrix $\sum_j \sum_m \langle j_1 m_1; j_2 m_2 | [j_1 j_2]; jm\rangle\langle j_1 m_1'; j_2 m_2' | [j_1 j_2] jm\rangle = \delta_{m_1 m_1'}\delta_{m_2 m_2'}$

& $\sum_{m_1}\sum_{m_2} \langle j_1 m_1; j_2 m_2 | [j_1 j_2]; jm\rangle\langle j_1 m_1; j_2 m_2 | [j_1 j_2] j'm'\rangle = \delta_{jj'}\delta_{mm'}$

The CGCs have a number of symmetry properties, e.g.

$$\langle j_1 m_1 ; j_2 m_2 | [j_1 j_2]; jm \rangle = (-1)^{j_1+j_2-j} \langle j_2 m_2 ; j_1 m_1 | [j_1 j_2]; jm \rangle$$

$$\langle j_1 m_1 ; j_2 m_2 | [j_1 j_2]; jm \rangle = (-1)^{j_1+j_2-j} \langle j_1, -m_1 ; j_2, -m_2 | [j_1 j_2]; j, -m \rangle$$

$$\langle j_1 m_1 ; j_2 m_2 | [j_1 j_2]; jm \rangle = (-1)^{j_2+m_2} \sqrt{\frac{2j+1}{2j_1+1}} \langle j_2 -m_2 ; jm | [j_2 j]; j_1 m_1 \rangle$$

There are several more - look in textbooks.

The easiest way to get the value of a CGC is to use tables, e.g. the Particle Data Group include CGC tables in their handbook : http://pdg.lbl.gov/2008/reviews/clebrpp.pdf

If you have access to Mathematica you can use the built in function "ClebschGordan"

$$\langle j_1 m_1 ; j_2 m_2 | [j_1 j_2]; jm \rangle = \text{ClebschGordan}[\{j_1, m_1\}, \{j_2, m_2\}, \{j, m\}]$$

In the homework you'll go through an exercise to derive some CGCs using ladder operators.

* an important example, coupling spin-½ to orbital angular momentum:

Suppose a spin-½ particle (e.g. an electron) is allowed to move in three-dimensions. We know that an acceptable basis is the set of eigenstates of $\hat{L}^2$, L_z, supplemented by something describing the radial state, $|n\ell m\rangle$

$$\langle \vec{x} | n\ell m\rangle = \langle r\theta\phi | n\ell m\rangle = f_n(r)\, Y_\ell^m(\theta,\phi)$$

then any state can be expressed $|\alpha\rangle = \sum_{n\ell m} \langle n\ell m|\alpha\rangle\, |n\ell m\rangle$

or explicitly in the position basis $\langle \vec{x}|\alpha\rangle = \psi_\alpha(\vec{x}) = \sum_{n\ell m} \langle \vec{x}|n\ell m\rangle\langle n\ell m|\alpha\rangle$

$$= \sum_{n\ell m} c^{(\alpha)}_{n\ell m}\, \underbrace{f_n(r)}_{\text{complete set of radial functions}}\, \underbrace{Y_\ell^m(\theta\phi)}_{\text{complete set of angular functions}}$$

Thus to describe a spin-½ particle moving in three-dimensions we can use the product basis

$$|n\ell m_\ell ; s=\tfrac{1}{2}, m_s = \pm\tfrac{1}{2}\rangle \equiv |n\ell m_\ell\rangle \otimes |s=\tfrac{1}{2}, m_s=\pm\tfrac{1}{2}\rangle$$

which, except for the radial quantum number, n, is just $(j_1 = \ell = \text{integer}) \otimes (j_2 = s = \tfrac{1}{2})$.

Hence we expect to be able to couple these to $j = \ell \pm \tfrac{1}{2}$ (for $\ell > 0$)

$$\left[\text{or } j = \tfrac{1}{2} \quad (\text{for } \ell = 0)\right]$$

e.g. $\ell = 1$ "P-wave"

* $j=\tfrac{1}{2}$: $|[1\tfrac{1}{2}];\tfrac{1}{2}m\rangle = \sum_{m_\ell m_s} \langle 1 m_\ell ; \tfrac{1}{2} m_s | [1\tfrac{1}{2}]; \tfrac{1}{2} m\rangle\, |1 m_\ell ; \tfrac{1}{2} m_s\rangle$

$$\hookrightarrow |[1\tfrac{1}{2}];\tfrac{1}{2},+\tfrac{1}{2}\rangle = \langle 1{+}1;\tfrac{1}{2}{-}\tfrac{1}{2}|[1\tfrac{1}{2}];\tfrac{1}{2}{+}\tfrac{1}{2}\rangle\cdot|1{+}1;\tfrac{1}{2},-\tfrac{1}{2}\rangle + \langle 10;\tfrac{1}{2}{+}\tfrac{1}{2}|[1\tfrac{1}{2}];\tfrac{1}{2}{+}\tfrac{1}{2}\rangle\cdot|10;\tfrac{1}{2}{+}\tfrac{1}{2}\rangle$$

$$|[1\tfrac{1}{2}];\tfrac{1}{2},+\tfrac{1}{2}\rangle = \sqrt{\tfrac{2}{3}}\,|1{+}1;\tfrac{1}{2}{-}\tfrac{1}{2}\rangle - \sqrt{\tfrac{1}{3}}\,|10;\tfrac{1}{2},+\tfrac{1}{2}\rangle$$

& similarly $|[1\tfrac{1}{2}];\tfrac{1}{2},-\tfrac{1}{2}\rangle = -\sqrt{\tfrac{2}{3}}\,|1{-}1;\tfrac{1}{2}{+}\tfrac{1}{2}\rangle + \sqrt{\tfrac{1}{3}}\,|10;\tfrac{1}{2},-\tfrac{1}{2}\rangle$

check: $\langle [1\tfrac{1}{2}];\tfrac{1}{2}{+}\tfrac{1}{2}|[1\tfrac{1}{2}];\tfrac{1}{2},+\tfrac{1}{2}\rangle = \tfrac{2}{3}\overbrace{\langle 1{+}1;\tfrac{1}{2}{-}\tfrac{1}{2}|1{+}1;\tfrac{1}{2}{-}\tfrac{1}{2}\rangle}^{1} - \tfrac{\sqrt{2}}{3}\overbrace{\langle 1{+}1;\tfrac{1}{2}{-}\tfrac{1}{2}|10;\tfrac{1}{2}{+}\tfrac{1}{2}\rangle}^{0} - \tfrac{\sqrt{2}}{3}\overbrace{\langle 10;\tfrac{1}{2}{+}\tfrac{1}{2}|1{+}1;\tfrac{1}{2}{-}\tfrac{1}{2}\rangle}^{0} + \tfrac{1}{3}\overbrace{\langle 10;\tfrac{1}{2}{+}\tfrac{1}{2}|10;\tfrac{1}{2}{+}\tfrac{1}{2}\rangle}^{1} = 1$ ✓

$* \; j = \frac{3}{2} \; : \quad |[1\tfrac{1}{2}]; \tfrac{3}{2} m\rangle = \sum_{m_l m_s} \langle 1 m_l ; \tfrac{1}{2} m_s | [1\tfrac{1}{2}]; \tfrac{3}{2} m \rangle \, |1 m_l ; \tfrac{1}{2} m_s\rangle$

$$|[1\tfrac{1}{2}]; \tfrac{3}{2}, +\tfrac{3}{2}\rangle = |1{+1}; \tfrac{1}{2}, +\tfrac{1}{2}\rangle$$

$$|[1\tfrac{1}{2}]; \tfrac{3}{2}, +\tfrac{1}{2}\rangle = \frac{1}{\sqrt{3}} |1{+1}; \tfrac{1}{2} {-\tfrac{1}{2}}\rangle + \sqrt{\tfrac{2}{3}} \, |10; \tfrac{1}{2} {+\tfrac{1}{2}}\rangle$$

$$|[1\tfrac{1}{2}]; \tfrac{3}{2}, -\tfrac{1}{2}\rangle = \sqrt{\tfrac{2}{3}} \, |10; \tfrac{1}{2} {-\tfrac{1}{2}}\rangle + \sqrt{\tfrac{1}{3}} \, |1{-1}; \tfrac{1}{2} {+\tfrac{1}{2}}\rangle$$

$$|[1\tfrac{1}{2}]; \tfrac{3}{2}, -\tfrac{3}{2}\rangle = |1{-1}; \tfrac{1}{2} {-\tfrac{1}{2}}\rangle$$

note orthogonality of e.g. $|[1\tfrac{1}{2}]; \tfrac{3}{2}, +\tfrac{1}{2}\rangle$ with $|[1\tfrac{1}{2}]; \tfrac{1}{2} +\tfrac{1}{2}\rangle$:

$$\langle [1\tfrac{1}{2}]; \tfrac{3}{2}, +\tfrac{1}{2} | [1\tfrac{1}{2}]; \tfrac{1}{2} +\tfrac{1}{2}\rangle = \left(\sqrt{\tfrac{1}{3}} \langle 1{+1}; \tfrac{1}{2} {-\tfrac{1}{2}}| + \sqrt{\tfrac{2}{3}} \langle 10; \tfrac{1}{2} {+\tfrac{1}{2}}| \right) \left(\sqrt{\tfrac{2}{3}} |1{+1}; \tfrac{1}{2} {-\tfrac{1}{2}}\rangle - \sqrt{\tfrac{1}{3}} |10; \tfrac{1}{2} {+\tfrac{1}{2}}\rangle \right)$$

$$= \frac{\sqrt{2}}{3} - \frac{\sqrt{2}}{3} = \underline{\underline{0}}.$$

We can see how one basis set might be more useful than the other by considering a spherically symmetric potential supplemented by a spin-orbit interaction :

$$V = V(r) + \nu_{so} \, \hat{\vec{L}} \cdot \hat{\vec{S}}$$

If there were just $V(r)$, either the $\vec{L} \otimes \vec{S}$ product basis or the total $\vec{J}$ basis would be equally good with (l, m_l, s, m_s) or (j, m, l, s) each being simultaneous good quantum numbers. However the addition of the spin-orbit term means that only the $(j m l s)$ set will suffice as good quantum numbers.

$$\hat{\vec{L}} \cdot \hat{\vec{S}} = \hat{L}_x \hat{S}_x + \hat{L}_y \hat{S}_y + \hat{L}_z \hat{S}_z = \tfrac{1}{2} \left(\hat{L}_+ \hat{S}_- + \hat{L}_- \hat{S}_+ \right) + \hat{L}_z \hat{S}_z$$

$$\Rightarrow \; \langle l m_l ; s m_s | \hat{\vec{L}} \cdot \hat{\vec{S}} | l m_l' ; s m_s' \rangle \quad \text{is not diagonal.}$$

but $\hat{\vec{L}} \cdot \hat{\vec{S}} = \frac{1}{2} \left((\hat{\vec{L}} + \hat{\vec{S}}) \cdot (\hat{\vec{L}} + \hat{\vec{S}}) - \hat{L}^2 - \hat{S}^2 \right) = \frac{1}{2} \left(\hat{J}^2 - \hat{L}^2 - \hat{S}^2 \right)$

so $\langle [ls] jm | \hat{\vec{L}} \cdot \hat{\vec{S}} | [l's'] j'm' \rangle = \frac{\hbar^2}{2} \left(j(j+1) - l(l+1) - s(s+1) \right) \delta_{ll'} \delta_{ss'} \delta_{jj'} \delta_{mm'}$

$\Rightarrow$ Hamiltonian is diagonal in this basis, & these are stationary states

The CGCs are also vital ingredients in the construction of the rotation operator for states constructed by angular momentum addition.

Recall that rotation by an angle ϕ about an axis $\hat{n}$ has an action on quantum states:

$$|\alpha\rangle_R = \mathcal{D}(R)|\alpha\rangle = \exp\left[-\tfrac{i}{\hbar}\hat{n}\cdot\hat{\vec{J}}\phi\right]|\alpha\rangle$$

we can obtain an explicit matrix representation by working in the eigenbasis of $\hat{J}^2, \hat{J}_z$:

$$\mathcal{D}^{(j)}_{m'm}(R) \equiv \langle jm'|\mathcal{D}(R)|jm\rangle$$

e.g. rotation about the y-axis for $j=\frac{1}{2}$:

$$D^{(1/2)}(R_y(\phi)) = \begin{bmatrix} \cos\phi/2 & -\sin\phi/2 \\ \sin\phi/2 & \cos\phi/2 \end{bmatrix}$$

see problem set 7 last semester.

Now suppose we consider a basis $\{|jm\rangle\}$ constructed by $j_1 \otimes j_2$.

- the rotation operator for the product states is

$$D_{(1)}(R) \otimes D_{(2)}(R) = \exp\left[-\tfrac{i}{\hbar}\hat{n}\cdot\hat{\vec{J}}_{(1)}\phi\right] \otimes \exp\left[-\tfrac{i}{\hbar}\hat{n}\cdot\hat{\vec{J}}_{(2)}\phi\right]$$

which in the basis $|j_1m_1;j_2m_2\rangle = |j_1m_1\rangle \otimes |j_2m_2\rangle$ can be expressed

$$D^{(j_1)}_{m_1'm_1}(R)\,D^{(j_2)}_{m_2'm_2}(R) \equiv \langle j_1m_1'|D(R)|j_1m_1\rangle\langle j_2m_2'|D(R)|j_2m_2\rangle. \quad (A)$$

- the rotation operator for the coupled states is $D(R) = \exp\left[-\tfrac{i}{\hbar}\hat{n}\cdot\hat{\vec{J}}\phi\right]$

which in the basis $|[j_1j_2];jm\rangle$ can be expressed

$$D^{(j)}_{m'm}(R) \equiv \langle jm'|D(R)|jm\rangle$$

but we can consider (A) to be $\langle j_1m_1';j_2m_2'|D(R)|j_1m_1;j_2m_2\rangle$

& since $|j_1m_1;j_2m_2\rangle = \sum_{j=|j_1-j_2|}^{j_1+j_2}\sum_m \langle[j_1j_2]jm|j_1m_1;j_2m_2\rangle\,|[j_1j_2]jm\rangle$

$$(A) \rightarrow \sum_{j=|j_1-j_2|}^{j_1+j_2}\sum_{mm'} \langle[j_1j_2]jm|j_1m_1;j_2m_2\rangle\langle[j_1j_2]jm'|j_1m_1';j_2m_2'\rangle \cdot \langle jm'|D(R)|jm\rangle$$

where we used the fact that a rotation cannot change the value of j.

$$\boxed{\text{\& thus}\quad D^{(j_1)}_{m_1'm_1}(R)\,D^{(j_2)}_{m_2'm_2}(R) = \sum_{j=|j_1-j_2|}^{j_1+j_2}\sum_{mm'} \langle[j_1j_2];jm|j_1m_1;j_2m_2\rangle\langle[j_1j_2]jm'|j_1m_1';j_2m_2'\rangle\, D^{(j)}_{m'm}(R)}$$

invert $D^{(j_1)}_{m_1'm_1} D^{(j_2)}_{m_2'm_2} = \sum_{j} \sum_{m,m'} \langle j_1m_1;j_2m_2|jm\rangle \langle j_1m_1';j_2m_2'|jm'\rangle D^{(j)}_{m'm}$

$$\sum_{\substack{m_1m_2\\m_1'm_2'}} \langle j_1\bar{m}_1;j_2m_2|\bar{j}\bar{m}\rangle\langle j_1m_1';j_2m_2'|\bar{\bar{j}}\bar{\bar{m}}\rangle D^{(j_1)}_{m_1'm_1} D^{(j_2)}_{m_2'm_2}$$

$$= \sum_{j}\sum_{mm'} \underbrace{\langle \bar{j}\bar{m}|jm\rangle}_{\delta_{\bar{j}j}\delta_{\bar{m}m}} \underbrace{\langle \bar{\bar{j}}\bar{\bar{m}}|jm'\rangle}_{\delta_{\bar{\bar{j}}j}\delta_{\bar{\bar{m}}m'}} D^{(j)}_{m'm}$$

$$= \delta_{\bar{j}\bar{\bar{j}}} D^{(j)}_{\bar{\bar{m}}m}$$

$$\Rightarrow \boxed{D^{(j)}_{\bar{\bar{m}}m} = \sum_{\substack{m_1m_2\\m_1'm_2'}} \langle j_1m_1;j_2m_2|jm\rangle\langle j_1m_1';j_2m_2'|jm'\rangle D^{(j_1)}_{m_1'm_1} D^{(j_2)}_{m_2'm_2}}$$