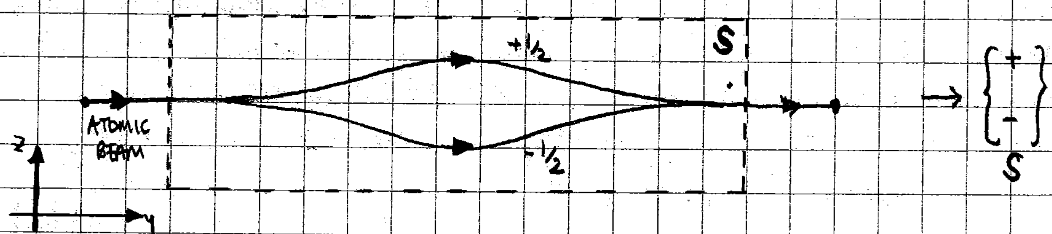
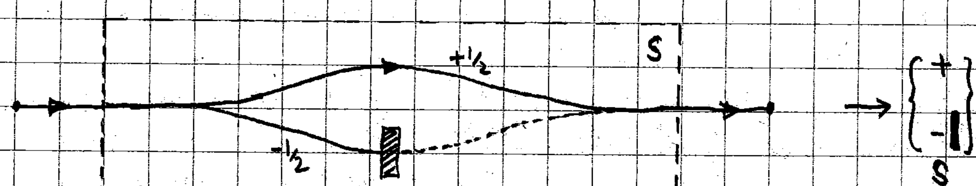
2. A TWO-STATE SYSTEM

Let's consider a different system from the world of atomic physics. We're going to think in terms of a device called a 'Stern-Gerlach' apparatus.
The details of how it works need not worry us here - all we need to know is that the device splits a beam of atoms into separate beams according to their quantum state. We'll consider atoms where there are two possible states, labelled $+\frac{1}{2}$ & $-\frac{1}{2}$ which eventually we'll identify with 'intrinsic spin'.
For now, consider that our device acts as follows:



We also have the ability to insert plates to block either of the beams



If we then put a second S apparatus, identical to the first after it, the beam would only follow the upper path. Thus our apparatus, complete with blocking plates can act as a filter.

Consider putting two filtering S-apparatus in series, how much of the beam present after the first S is still present after the second?

[1] $\left\{\begin{matrix}+\\ -\blacksquare\end{matrix}\right\}_S \left\{\begin{matrix}+\\ -\blacksquare\end{matrix}\right\}_S \to$ 100% transmission

[2] $\left\{\begin{matrix}+\\ -\blacksquare\end{matrix}\right\}_S \left\{\begin{matrix}+\blacksquare\\ -\end{matrix}\right\}_S \to$ 0% transmission

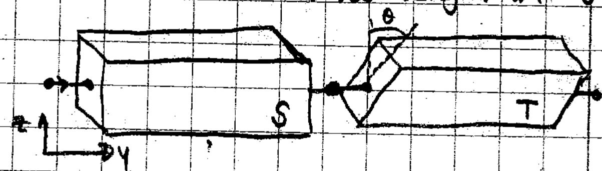
[3] $\left\{\begin{matrix}+\blacksquare\\ -\end{matrix}\right\}_S \left\{\begin{matrix}+\blacksquare\\ -\end{matrix}\right\}_S \to$ 100% transmission

[4] $\left\{\begin{matrix}+\blacksquare\\ -\end{matrix}\right\}_S \left\{\begin{matrix}+\\ -\blacksquare\end{matrix}\right\} \to$ 0% transmission

Let's try to describe these experiments in terms of quantum-mechanical amplitudes. We'll define the state of an atom after it has passed through $\left\{\begin{matrix}+\\ -\blacksquare\end{matrix}\right\}$ by $|S+\rangle$ & $\left\{\begin{matrix}+\blacksquare\\ -\end{matrix}\right\}$ by $|S-\rangle$.

Amplitude for experiment [1] $= \langle S+|S+\rangle = 1$; [3] $\langle S-|S-\rangle = 1$
[2] $= \langle S-|S+\rangle = 0$; [4] $\langle S+|S-\rangle = 0$

Nothing too Earth-shattering so far. What if we rotate the second apparatus about the y-axis by some angle θ? Since it is no longer an 'S' apparatus we'll now label it T.



T can also have blocking plates inserted. Suppose we set up the expt

$$\begin{array}{cc}\begin{Bmatrix}+\\-\end{Bmatrix} & \begin{Bmatrix}+\\-\end{Bmatrix}\\ S & T\end{array}$$

, what transmission do we get?

The answer is - not necessarily 100%! The T-filter lets through only $|T+\rangle$ states, but $|T+\rangle$ states are not necessarily the same as $|S+\rangle$ states. We have to know amplitudes like $\langle T+|S+\rangle$ in order to work out the transmission. In fact total knowledge corresponds to four complex numbers

$$\langle T+|S+\rangle \quad \langle T+|S-\rangle$$
$$\langle T-|S+\rangle \quad \langle T-|S-\rangle.$$

We can infer a little about these numbers. The probability that an S+ state will enter a T+ state is

$$|\langle T+|S+\rangle|^2 = \langle T+|S+\rangle(\langle T+|S+\rangle)^*$$

similarly the probability an S+ state will enter a T- state is

$$|\langle T-|S+\rangle|^2 = \langle T-|S+\rangle(\langle T-|S+\rangle)^*,$$

But every atom that enters in an S+ state must become either a T+ or T- state - there are no other options, so the sum of the two probabilities above had better be 1,

$$\underline{\langle T+|S+\rangle(\langle T+|S+\rangle)^* + \langle T-|S+\rangle(\langle T-|S+\rangle)^* = 1} \quad \text{Ⓐ}$$

equivalently for S- states:

$$\underline{\langle T+|S-\rangle(\langle T+|S-\rangle)^* + \langle T-|S-\rangle(\langle T-|S-\rangle)^* = 1}$$

Now suppose we have three apparatus in series:

[1] $\underset{S}{\left\{\begin{matrix} + \\ -| \end{matrix}\right\}} \quad \underset{T}{\left\{\begin{matrix} +| \\ - \end{matrix}\right\}} \quad \underset{S'}{\left\{\begin{matrix} + \\ -| \end{matrix}\right\}}$ (S' is aligned the same as S)

We might wonder if the atoms arriving at S' 'remember' that they were once in an $S+$ state? The answer is that they do not – if the T-filter passes just one beam, the fraction passing through S' depends only on T and not at all on S or anything beforehand.

For example, compare the above experiment with the following:

[2] $\underset{S}{\left\{\begin{matrix} + \\ -| \end{matrix}\right\}} \quad \underset{T}{\left\{\begin{matrix} +| \\ - \end{matrix}\right\}} \quad \underset{S'}{\left\{\begin{matrix} +| \\ - \end{matrix}\right\}}$

[1]: $\langle S'+|T-\rangle\langle T-|S+\rangle \Rightarrow P_{[1]} = |\langle S'+|T-\rangle|^2\,|\langle T-|S+\rangle|^2$

[2]: $\langle S'-|T-\rangle\langle T-|S+\rangle \Rightarrow P_{[2]} = |\langle S'-|T-\rangle|^2\,|\langle T-|S+\rangle|^2$

ratio of transmission $= \dfrac{P_{[2]}}{P_{[1]}} = \dfrac{|\langle S'+|T-\rangle|^2}{|\langle S'-|T-\rangle|^2}$ which is independent of the state selected by S.

The states, for which the previous history has no effect after a filter are called 'base states' or 'basis states' and they depend upon the filter used. For S they were $|S+\rangle, |S-\rangle$, for T they were $|T+\rangle, |T-\rangle$.

It is the 'blocking' that gives rise to this 'loss of memory', it constitutes a 'measurement' of which basis state the atom is in. We can check that it is the blocking an not the 'splitting' of the beams as follows:

[1] $\underset{S}{\left\{\begin{matrix} + \\ -| \end{matrix}\right\}} \xrightarrow{N} \underset{T}{\left\{\begin{matrix} +| \\ - \end{matrix}\right\}} \xrightarrow{fN} \underset{S'}{\left\{\begin{matrix} + \\ -| \end{matrix}\right\}} \xrightarrow{ff'N}$ / [2] $\underset{S}{\left\{\begin{matrix} + \\ -| \end{matrix}\right\}} \xrightarrow{N} \underset{T}{\left\{\begin{matrix} +| \\ - \end{matrix}\right\}} \xrightarrow{fN} \underset{S'}{\left\{\begin{matrix} +| \\ - \end{matrix}\right\}} \xrightarrow{ff''N}$

the fractions are all <1, so the blocking reduces the transmission

now repeat but without any blocking in T

[3] $\underset{S}{\left\{\begin{matrix} + \\ -| \end{matrix}\right\}} \xrightarrow{N} \underset{T}{\left\{\begin{matrix} + \\ - \end{matrix}\right\}} \xrightarrow{N} \underset{S'}{\left\{\begin{matrix} + \\ -| \end{matrix}\right\}} \xrightarrow{N}$ / [4] $\underset{S}{\left\{\begin{matrix} + \\ -| \end{matrix}\right\}} \xrightarrow{N} \underset{T}{\left\{\begin{matrix} + \\ - \end{matrix}\right\}} \xrightarrow{N} \left\{\begin{matrix} +| \\ - \end{matrix}\right\} \xrightarrow{0}$

Compare [2] & [4] – in [4] we opened up <u>more</u> channels and <u>fewer</u> atoms got through! This is distinctly non-classical behaviour.

Consider [4] in the language of amplitudes: $\langle S-|T+\rangle\langle T+|S+\rangle + \langle S-|T-\rangle\langle T-|S+\rangle = 0$

neither of the terms is zero, but the sum is. This is interference.

e.g. the second term features in [2].

what about [3]? In amplitudes we'd write this

$$\langle S+|T+\rangle\langle T+|S+\rangle + \langle S+|T-\rangle\langle T-|S+\rangle = 1$$

From the last two equations we see that it is as though T where not there at all

$$\sum_i \langle S-|i\rangle\langle i|S+\rangle = 0 \quad = \langle S-|S+\rangle$$

$$\sum_i \langle S+|i\rangle\langle i|S+\rangle = 1 \quad = \langle S+|S+\rangle$$

$$\left(\sum_i |i\rangle\langle i| = 1\right)$$

In general we need not use another S apparatus as the third in the series, we could use an apparatus of any alignment, call it R, then $\underset{S}{\left\{ {+ \atop -} \right\}} \underset{T}{\left\{ {+ \atop -} \right\}} \underset{R}{\left\{ {+ \atop -} \right\}}$

$$\sum_i \langle R+|i\rangle\langle i|S+\rangle = \langle R+|S+\rangle$$

In complete generality, for any initial state $|\phi\rangle$ and any final state $\langle\chi|$, and a complete set of basis states $|i\rangle$ we have

$$\boxed{\langle\chi|\phi\rangle = \sum_i \langle\chi|i\rangle\langle i|\phi\rangle} \quad (B)$$

In our case of S, T, S' filters, the equation above is

$$\underset{1}{\underset{\parallel}{\langle S+|S+\rangle}} = \langle S+|T+\rangle\langle T+|S+\rangle + \langle S+|T-\rangle\langle T-|S+\rangle$$

but recall that conserving probability led us to (A):

$$1 = (\langle T+|S+\rangle)^* \langle T+|S+\rangle + (\langle T-|S+\rangle)^* \langle T-|S+\rangle$$

these two equations can only be satisfied if $\langle S+|T+\rangle = (\langle T+|S+\rangle)^*$ etc...
or in complete generality $\boxed{\langle\phi|\chi\rangle = \langle\chi|\phi\rangle^*}$

Extending the machinery: suppose we start with an S-filter, finish with an R-filter an put 'anything' between them.

$$\begin{Bmatrix} + \\ -| \end{Bmatrix}_S \begin{Bmatrix} A \end{Bmatrix} \begin{Bmatrix} +| \\ - \end{Bmatrix}_R$$

'anything' can be any combination of apparatus, that filters, or doesn't ...

We extend our amplitude notation a little and write this

$\langle R- | A | S+ \rangle$ → start in S+, 'apply' A, end in R−.

or in more generality, for arbitrary start & end states

$$\langle \chi | A | \phi \rangle .$$

To completely describe the effect of A it appears we need to know $\langle \chi | A | \phi \rangle$ for any possible $|\phi\rangle$ & $\langle \chi|$ - well that's an infinite number of numbers!

Suppose though that we insert some "non-block" T apparatus either side of A:

$$\begin{Bmatrix} + \\ -| \end{Bmatrix}_S \begin{Bmatrix} + \\ - \end{Bmatrix}_T \begin{Bmatrix} A \end{Bmatrix} \begin{Bmatrix} + \\ - \end{Bmatrix}_T \begin{Bmatrix} +| \\ - \end{Bmatrix}_R$$

. Of course this hasn't done anything since the T's let everything through, but look at the form of the amplitude:

$$\langle R- | A | S+ \rangle = \sum_{ij} \langle R- | j \rangle \langle j | A | i \rangle \langle i | S+ \rangle$$

or generally

$$\langle \chi | A | \phi \rangle = \sum_{ij} \langle \chi | j \rangle \langle j | A | i \rangle \langle i | \phi \rangle$$

Now this is not so bad, there are only two basis states of T: $|i\rangle = |T+\rangle, |T-\rangle$ so A can be described by only four complex numbers.

We'll see that representing an 'operator' like A in a particular basis will offer us a way to deal with the mathematics of quantum mechanics.

There is one point to be made here - what basis should we use?

Say Sacha chose to use the basis (S+, S−) and Taika chose to use the basis (T+, T−), how would their results for a measurement differ? They wouldn't, but their description of states in the calculation wouldn't be exactly the same. They are simply related though

e.g. Taika describes an arbitrary state $|\phi\rangle$ by two numbers, $\langle T_j | \phi \rangle$ and Sacha uses $\langle S_i | \phi \rangle$

Using eqn (B): $\langle T_j | \phi \rangle = \sum_i \langle T_j | S_i \rangle \langle S_i | \phi \rangle$

So the transformation from the S basis to T requires the numbers $\langle T_j | S_i \rangle$.

I'll state without proof here the transformation from S to T, where T is rotated about $\hat{y}$ by θ (we'll derive this later on)

$$\langle T+|S+\rangle = \cos\frac{\theta}{2} \qquad \langle T+|S-\rangle = \sin\theta/2$$
$$\langle T-|S+\rangle = -\sin\theta/2 \qquad \langle T+|S-\rangle = \cos\theta/2$$

Then we can explicitly express say $|+T\rangle$ in terms of $|+S\rangle, |-S\rangle$

$$\langle T_j|\phi\rangle = \sum_i \langle T_j|S_i\rangle\langle S_i|\phi\rangle$$

complex conjugate both sides: $\langle\phi|T_j\rangle = \sum_i \langle\phi|S_i\rangle\langle S_i|T_j\rangle$
this is true for all $\langle\phi|$ so

$$|T_j\rangle = \sum_i |S_i\rangle\langle S_i|T_j\rangle$$

e.g. $|T+\rangle = \sum_i |S_i\rangle\langle S_i|T+\rangle = |S+\rangle\langle S+|T+\rangle + |S-\rangle\langle S-|T+\rangle$

$$\underline{|T+\rangle = \cos\theta/2\,|S+\rangle + \sin\theta/2\,|S-\rangle}$$

similarly $\underline{|T-\rangle = -\sin\theta/2\,|S+\rangle + \cos\theta/2\,|S-\rangle}$

note that $\langle T+|T+\rangle = \cos^2\theta/2\,\langle S+|S+\rangle + \cos\theta/2\sin\theta/2\,\langle S+|S-\rangle + \cos\theta/2\sin\theta/2\,\langle S-|S+\rangle + \sin^2\theta/2\,\langle S+|S+\rangle$

$$= \cos^2\theta/2 + \sin^2\theta/2 = 1$$

$$\langle T+|T-\rangle = 0$$

as we demand of basis states.

We might consider changing our apparatus so that rather than blocking a separated beam, for each atom passing through the device we measure whether it took the upper or lower path, returning $+\frac{1}{2}$ for the upper path and $-\frac{1}{2}$ for the lower path. This is a measurement device that we'll represent by the <u>operator</u> S. Then clearly

$$\langle S+|S|S+\rangle = +\tfrac{1}{2}$$
$$\langle S+|S|S-\rangle = 0$$
$$\langle S-|S|S+\rangle = 0$$
$$\langle S-|S|S-\rangle = -\tfrac{1}{2}$$

alternatively we can say

$$S|S+\rangle = +\tfrac{1}{2}|S+\rangle$$
$$S|S-\rangle = -\tfrac{1}{2}|S-\rangle$$

We call S an 'operator' because it operates on a state to produce a new state.

Note that if we put the basis-states of T; $|T+\rangle, |T-\rangle$ into an S measurement device the situation is not so simple:

$$S\,|T+\rangle = S\left(\cos\tfrac{\theta}{2}\,|S+\rangle + \sin\tfrac{\theta}{2}\,|S-\rangle\right) = +\tfrac{1}{2}\cos\tfrac{\theta}{2}\,|S+\rangle - \tfrac{1}{2}\sin\tfrac{\theta}{2}\,|S-\rangle$$

$$= \tfrac{1}{2}\left(\cos^2\tfrac{\theta}{2} - \sin^2\tfrac{\theta}{2}\right)|T+\rangle - \tfrac{1}{2}\cos\tfrac{\theta}{2}\sin\tfrac{\theta}{2}\,|T-\rangle$$

Now that we see how amplitudes & basis states enter in a simple two state system we will break away from a specific example to consider the general mathematical properties of states & amplitudes.