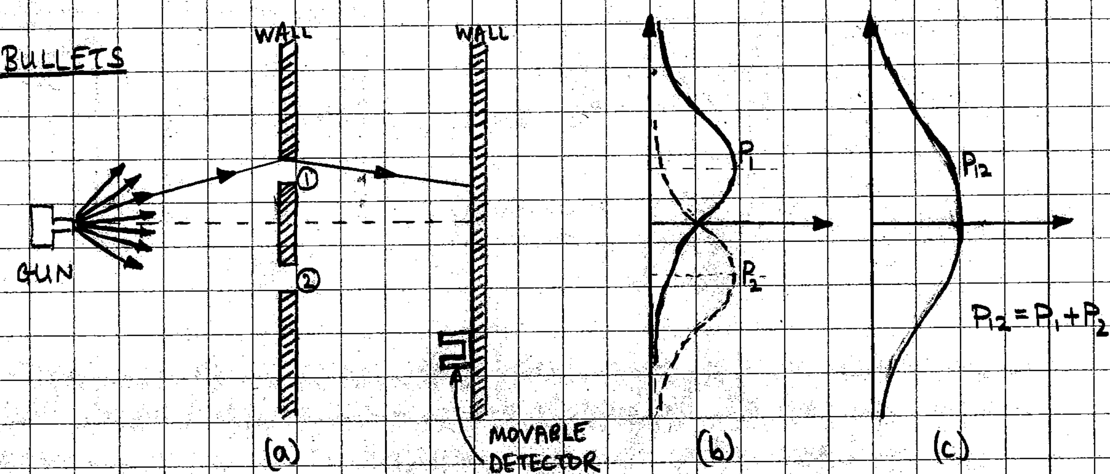
1. QUANTUM BEHAVIOUR, INTERFERENCE & PROBABILITY AMPLITUDES

The need for something beyond classical physics can be demonstrated by means of a simple 'thought' experiment. It is a 'thought' experiment in the sense that it would be basically impossible to fabricate the arrangement required, but the results will be those expected by the theory we'll develop which also describes a lot of real experiments.

The experiment we'll consider is one you're probably familiar with – a 'two-slits' setup. We'll consider this expt with three different 'projectiles': bullets, water waves & electrons.

* BULLETS



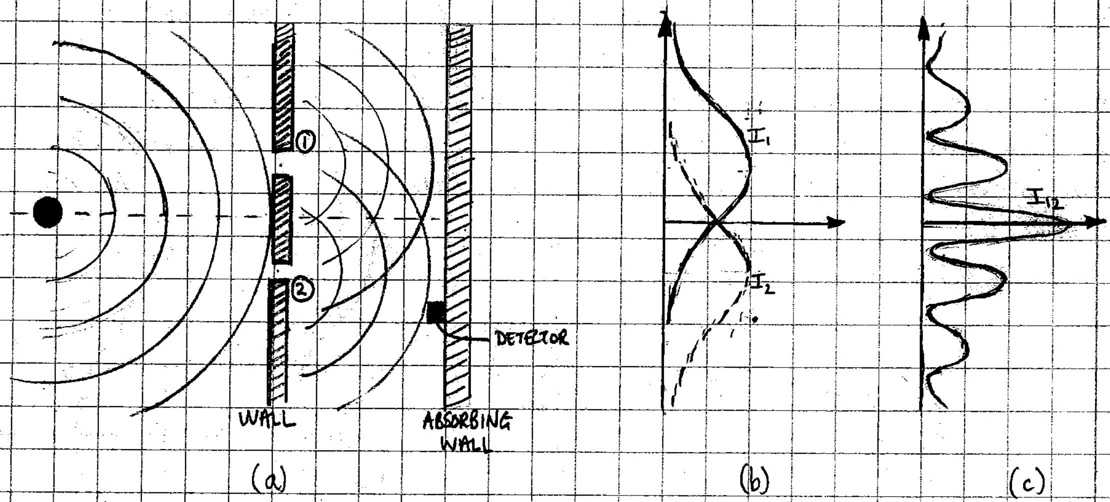
Gun sprays bullets over a wide angular spread. Many of them are 'absorbed' by the first wall, but some make it through the small holes in the wall. We place a detector on a more distant wall and, with the detector at various positions along the wall, count the number of bullets detected in a fixed period of time. This is proportional to the probability for a bullet to arrive at each position.
If we slow the firing rate of the gun we'll find that the detector always detects one or no bullets – as is obvious to us, bullets come in discrete lumps.
What we get if we perform this expt is the graph shown above as (c). We call the probability distn P_{12} since each bullet went through either hole 1 or hole 2.
Now what happens if we block off hole 2? Clearly we expect fewer bullets to arrive in the lower half of the diagram – the distn is P_1 as shown in (b). Similarly if we cover hole 1 we get P_2.
When we look closely at P_{12}, P_1 and P_2 we find that

$$P_{12} = P_1 + P_2$$ – the probabilities add.

Nothing unexpected here – this is what we think of as the behaviour of particles.

* WATER WAVES

Now let's set up a similar experiment with walls in a shallow water-filled tank. A bob is wiggled up and down by a motor producing waves of fixed frequency & wavelength.



Now we have a detector which measures the height of the wave and squares it so we can read off the 'intensity' of the wave at that point. Note that this can take any value from zero upwards - there is no discreteness or 'lumpiness' to the wave.

If we move the detector along the wall taking measurements of the intensity we get the graph shown above as (c). If we close either hole 1 or hole 2 we get the curves shown in (b). It is clearly that I_{12} is not just the sum of I_1 & I_2 - there is 'interference'.

The explanation is quite simple, if the path difference to the detector is an integer number of wavelengths, the waves arrive 'in phase' and add constructively. If the path difference is $(n+\frac{1}{2})\lambda$ then the waves are 'out of phase' and the wave amplitudes sum to zero.

[for small holes separated by a distance d, there are zeroes when $\sin\theta = (n+\frac{1}{2})\frac{\lambda}{d}$]

For interfering waves, we <u>add the amplitudes</u> : $I_1 = |h_1|^2$; $I_2 = |h_2|^2$

$$h_{12} = h_1 + h_2$$

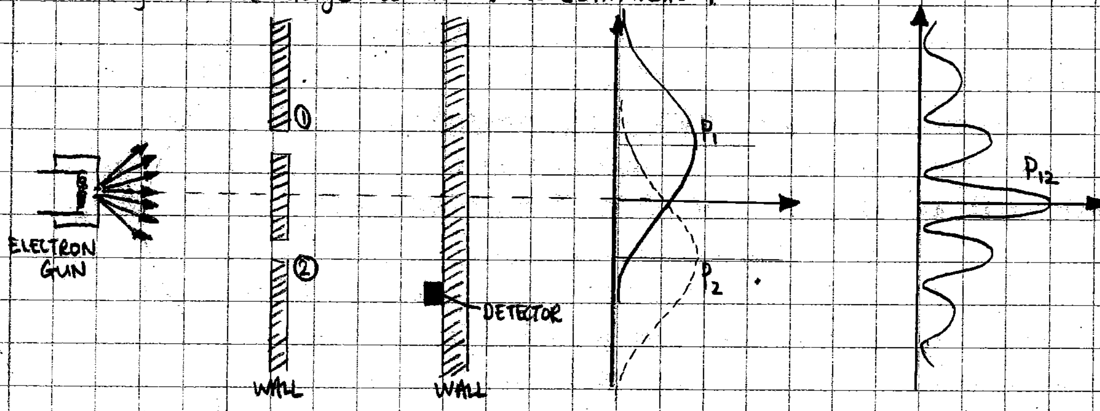
$$I_{12} = |h_1 + h_2|^2 = |h_1|^2 + |h_2|^2 + 2|h_1||h_2|\cos\delta$$

$$I_{12} = I_1 + I_2 + 2\sqrt{I_1 I_2}\cos\delta$$

This is what we've come to expect of waves in classical physics - they have a continuous intensity & they show interference effects.

X ELECTRONS

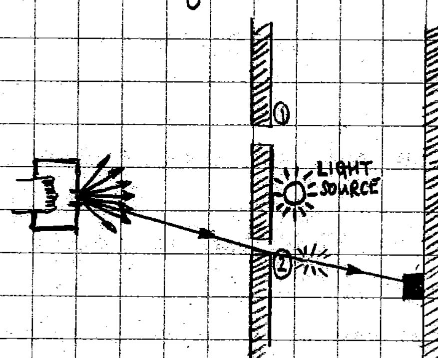
OK, so much for the classical experiments with which we define 'particle-behaviour' & 'wave-behaviour'. Now let's see what happens if we perform the experiment with electrons. Assume we can make an electron gun that produces electrons all with roughly the same energy. We'll place this in front of our hole-wall set up. For a detector we use something like a Geiger counter or a scintillator.



(a) (b) (c)

In this experiment we would notice the detector behaving just like the bullet detector, either it fires or it doesn't. The electrons arrive at the detector like discrete particles. As we move the detector up and down the wall, counting the number of hits in a fixed time, we plot out a 'probability' curve like that shown in (c). This should be a surprise if all you know is classical physics! The detector detects discrete particles, but the distribution of those particles is 'wave-like' – it clearly shows interference.

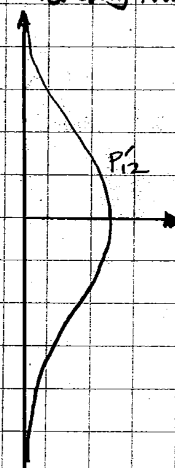
If we cover either hole 1 or hole 2 we get the distribution shown in (b), which appear to be quite reasonable. It looks like although the electrons arrive at the detector like particles, they traverse the experiment like waves! In that case you can't say which hole any particular electron went through. Let's test this idea further by 'watching' the electrons:



The addition of a light source allows us to track the electrons, since the light will scatter off the electron and we'll see a 'flash' near either 1 or 2. What find is exactly that, either a flash near 1 or near 2, indicating the electron goes through either 1 or 2!

This is pretty odd, but while we have the apparatus set up we can do a little more with it:

Every time the detector registers an electron we write down if we saw a flash near 1 or a flash near 2. After we've moved the detector up & down we can plot the "flash near 1" probability distribution. It looks basically the same as P_1 in (b) above. Similarly, the "flash near 2" distribution looks like P_2. This looks fine - electrons coming through hole 1 are distributed as P_1 - just like for bullets.



But now suppose we plot the total probability, that is we ignore the information about the flashes & just plot all electrons arriving. The distribution looks like the one to the right. ($P'_{12} = P'_1 + P'_2$)

Now this should be a shock - the distribution of electrons with a light source present is different to that without a light source. When we detect which hole the electrons went through we destroy the interference.

So it appears that the light source is disturbing the electrons in some way - perhaps we can lessen the effect by using a less-bright source, ie by lowering the intensity, so that the electrons aren't "hit" so hard? If we do this we find that we still see the same 'sized' flash, but sometimes we don't see a flash when an electron passes. This can be understood if the light is particle-like in behaviour - lowering the intensity means fewer particles, but each one having the same momentum.

Say we list
- all detector hits with a flash near 1 : LIST 1
- all detector hits with a flash near 2 : LIST 2
- all detector hits with no flash : LIST 3

and we plot the probability distr^ns. We find

LIST 1 $\rightarrow P_1$
LIST 2 $\rightarrow P_2$; LIST 1 + LIST 2 $\rightarrow P'_{12}$ (no interference) ; LIST 3 $\rightarrow P_{12}$ (interference)

So if the electron's path is not detected we get back the interference.

What happens if instead of lowering the intensity we raise the wavelength? You might know that the momentum of a photon is $p = h/\lambda$, so this will lower the momentum and give the electron a smaller 'kick'. Perhaps then it won't disturb the interference?

What is found is that as the wavelength of the light increases, the electron distribution starts to get more wave-like. But, there is a side effect - as the wavelength increases, the resolution decreases and instead of a clear flash near 1 or 2 we have a fuzzy flash from which we can't tell if the electron passed through 1 or 2 !?

It is generally true that there is no way to determine which hole the electron went through without destroying the interference. This is known as the <u>uncertainty</u> <u>principle</u> and is due to Heisenberg.

We can summarise what we've seen as:

[1] the probability of an event is given by the square of a complex number called the 'probability amplitude' $P = |\phi|^2$

[2] when an event can occur in several alternative ways the probability amplitude is the sum of the amplitudes for each way considered separately : $\phi = \phi_1 + \phi_2 + \dots$

$$P = |\phi_1 + \phi_2 + \dots|^2$$

[3] if an experiment is performed which is capable of determining which alternative is actually taken, then the probabilities are summed

$$P = P_1 + P_2 + \dots = |\phi_1|^2 + |\phi_2|^2 + \dots$$

The two-slit experiment in terms of probability amplitudes

Let's write the probability amplitude for an electron to start at the source (s) and arrive at the position x along the back wall in the following way:

$$\langle x|s\rangle \sim \text{"particle leaves } s \text{ \& arrives at } x\text{"}$$

The probability amplitude is a single complex number.
The probability that a particle leaving s arrives at x is given by the absolute square of this amplitude

$$P_{xs} = |\langle x|s\rangle|^2 .$$

We decided that if there are two ways for the electron to get from s to x, that we should add the amplitudes for the two ways, i.e.

$$\langle x|s\rangle_{\text{both holes}} = \langle x|s\rangle_{\text{thru 1}} + \langle x|s\rangle_{\text{thru 2}} .$$

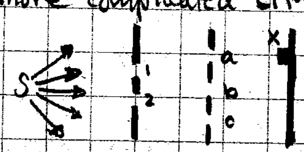
Now we propose a further rule, that when a particle goes by a certain route we multiply the amplitudes for some part of the route by the amplitude for the remainder, i.e.

$$\text{amplitude for } s\rightarrow 1 = \langle 1|s\rangle$$
$$\text{amplitude for } 1\rightarrow x = \langle x|1\rangle$$

$$\Rightarrow \text{amplitude for } s\rightarrow 1\rightarrow x = \langle x|1\rangle\langle 1|s\rangle$$

thus $\langle x|s\rangle_{\text{both holes}} = \langle x|1\rangle\langle 1|s\rangle + \langle x|2\rangle\langle 2|s\rangle \quad = \phi_1 + \phi_2$ in our previous notation.

These rules allow us to analyse more complicated situations in a straightforward manner, e.g.

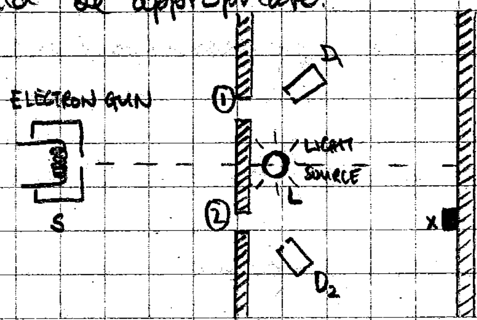


the complete amplitude is

$$\langle x|s\rangle = \langle x|a\rangle\langle a|1\rangle\langle 1|s\rangle + \langle x|b\rangle\langle b|1\rangle\langle 1|s\rangle + \langle x|c\rangle\langle c|1\rangle\langle 1|s\rangle$$
$$+ \langle x|a\rangle\langle a|2\rangle\langle 2|s\rangle + \langle x|b\rangle\langle b|2\rangle\langle 2|s\rangle + \langle x|c\rangle\langle c|2\rangle\langle 2|s\rangle$$
$$= \sum_{i=1}^{2}\sum_{j=a}^{c} \langle x|j\rangle\langle j|i\rangle\langle i|s\rangle$$

to actually compute the electron distribution we need to know how the amplitudes depend upon the distances between the slits etc... — we won't go into that much detail quite yet.

Recall that we were able to change the degree of interference in the electron distribution by attempting to "watch" the electrons as they passed through the slits. A set-up like the one below would be appropriate:



D_1 & D_2 are light detectors designed to catch photons scattered off electrons.

say that when an electron passed through 1 there is a probability amplitude, a, that light from L is scattered into D_1. Then the amplitude for

electron through 1 to x & light from L to D_1 is $= \langle x|1\rangle a \langle 1|s\rangle = a\phi_1$

now say that the light is long enough wavelength at there is a possibility for light to scatter off an electron at 2, but be detected by D_1 - it might be a small amplitude, but let's include it for generality. Then the amplitude for

electron through 2 to x & light from L to D_1 is $= \langle x|2\rangle b \langle 2|s\rangle = b\phi_2$

then the amplitude for

$$\left\langle \begin{matrix} \text{electron at } x \\ \text{photon in } D_1 \end{matrix} \middle| \begin{matrix} \text{electron at } s \\ \text{photon from } L \end{matrix} \right\rangle = a\phi_1 + b\phi_2$$

similarly, assuming the setup is symmetric

$$\left\langle \begin{matrix} \text{electron at } x \\ \text{photon in } D_2 \end{matrix} \middle| \begin{matrix} \text{electron at } s \\ \text{photon from } L \end{matrix} \right\rangle = a\phi_2 + b\phi_1 .$$

Suppose we count all events in which D_1 detects a photon. The probability for this is

$$P = |a\phi_1 + b\phi_2|^2.$$

→ Suppose further that the setup is such that the light resolves each hole clearly, then $\underline{b \to 0}$ &

$$P \to |a|^2 |\phi_1|^2$$

which is a single 'bump' peaked behind hole 1 – no interference.

→ Suppose that rather than this we had very long wavelength light so that the two holes cannot be resolved by the light, then $\underline{a \simeq b}$ &

$$P \to |a|^2 |\phi_1 + \phi_2|^2$$

which is an interfering pattern, just as if there was no detector.

We see that expression in terms of probability amplitudes can describe exactly what we found in the experiment.

CAUTION: what if you wanted the amplitude for an electron arriving at x, regardless of whether detector D_1 fired or D_2? You might be tempted to add:

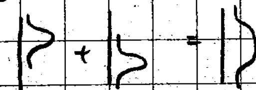
$$\left\langle \begin{matrix}\text{electron at } x \\ \text{photon in } D_1\end{matrix} \middle| \begin{matrix}\text{electron at } s \\ \text{photon from } L\end{matrix} \right\rangle + \left\langle \begin{matrix}\text{electron at } x \\ \text{photon in } D_2\end{matrix} \middle| \begin{matrix}\text{electron at } s \\ \text{photon from } L\end{matrix} \right\rangle.$$

But this would be wrong! These are distinct final states where you can, in principle, tell which happened. You only add amplitudes when you can't tell what happened in an intermediate state. Here since the final states are 'distinguishable' you must add the probabilities

$$\left|\left\langle \begin{matrix}\text{electron at } x \\ \text{photon in } D_1\end{matrix} \middle| \begin{matrix}\text{electron at } s \\ \text{photon from } L\end{matrix} \right\rangle\right|^2 + \left|\left\langle \begin{matrix}\text{electron at } x \\ \text{photon in } D_2\end{matrix} \middle| \begin{matrix}\text{electron at } s \\ \text{photon from } L\end{matrix} \right\rangle\right|^2$$

$$= |a\phi_1 + b\phi_2|^2 + |b\phi_1 + a\phi_2|^2$$

eg. if $b \to 0$ $\to |a|^2 \left(|\phi_1|^2 + |\phi_2|^2 \right)$



no interference