

Coefficient Stability of Three-Level Operator-Difference Schemes

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Abstract—Three-level operator-difference schemes with weights in Hilbert spaces are considered. A priori stability estimates are obtained for perturbations imposed on the operator (coefficients) of the Cauchy problem, on the initial conditions, and on the right-hand side of the equation. Coefficient stability in the L_2 norm is analyzed by invoking the superposition principle, which makes it possible to discard the unnatural condition of ρ -stability. Strong stability estimates are also presented in stronger norms.

1. INTRODUCTION

Two- and three-level schemes are most frequently used for time discretization in an approximate solution of time-dependent problems in mathematical physics [1]. In the analysis of the well-posedness of such schemes, primary attention is usually given to the stability of solutions with respect to the initial data and the right-hand side of the equation [1, 2]. However, in a numerical solution of a differential problem, the coefficients of the equation may be given approximately rather than exactly. Thus, it is important to analyze schemes with perturbed coefficients.

A difference scheme is called coefficient-stable (co-stable) if its solution changes little under small perturbations in its coefficients [1]. A priori estimates that express a continuous dependence of the solution on perturbations in the right-hand side and the operator of the equation (the so-called strong stability) were derived in [1–4] for time-independent problems. As to time-dependent problems, we note paper [5]. The strong stability of two-level operator-difference schemes was analyzed in [2, 4, 6–9].

We obtain strong stability estimates for three-level operator-difference schemes in Hilbert spaces.

2. AUXILIARY RESULTS

Suppose that H is a finite-dimensional real space and $\bar{\omega}_\tau = \{t_n = n\tau, n = 0, 1, \dots, N_0; N_0\tau = T\} = \omega_\tau \cup \{T\}$ is a given temporal grid. Let $A, B, D: H \rightarrow H$ denote linear operators defined in H and independent of τ or t_n .

Consider the Cauchy problem for the operator-difference equation

$$Dy_{i+1} + By_i + Ay = \varphi, \quad y_0 = u_0, \quad y_1 = u_1, \quad (2.1)$$

where $y_n = y(t_n) \in H$ is the desired function and $\varphi_n, u_0, u_1 \in H$ are given.

Hereafter, we use the standard index-free notation employed in [1, 2]: $y = y_n, \hat{y} = y_{n+1}, \check{y} = y_{n-1}, y_i = (y - \check{y})/\tau, y_i = (\hat{y} - y)/\tau, y_i = (\hat{y} - \check{y})/2\tau$, and $y_{ii} = (\hat{y} - 2y - \check{y})/\tau^2$. Denote by H_R (where $R^* = R > 0$) the space with the inner product $(y, v)_R = (Ry, v)$ and the norm $\|y\|_R^2 = (Ry, y)$.

Lemma 1 (see [2, p. 101]). *Let the operators A and D in (2.1) be constant and such that $A = A^* > 0$ and $D = D^* > 0.25\tau^2 A$. Then, $B \geq 0$ is a necessary and sufficient condition for scheme (2.1), with $\varphi = 0$, to be stable with respect to the initial data in $H_{\mathcal{A}}^2$; i.e., for any $\tau > 0$,*

$$\|Y_n\|_{\mathcal{A}} \leq \|Y_1\|_{\mathcal{A}}, \quad n = 1, 2, \dots, N_0, \quad (2.2)$$

where $Y_n = \{Y_n^{(1)}, Y_n^{(2)}\}$, $Y_n^{(1)} = 0.5(y_n + y_{n-1})$, $Y_n^{(2)} = y_n - y_{n-1}$, and $\|Y_n\|_{\mathcal{A}}^2 = 0.25\|y + \check{y}\|_A^2 + \|y_i\|_{D-0.25\tau^2 A}^2$.

Lemma 2 (see [2, p. 106]). *Let the operators A and D in (2.1) be such that*

$$A^* = A > 0, \quad D^* = D \geq 0.25(1 + \varepsilon)\tau^2 A, \quad \varepsilon > 0. \tag{2.3}$$

Then,

$$\sqrt{\frac{\varepsilon}{1 + \varepsilon}} \|y\|_A \leq \|Y\|_{\mathcal{A}} \leq \|\tilde{y}\|_A + \|y_i\|_D.$$

Lemma 3 (see [2, p. 125]). *Let conditions (2.3) be fulfilled, and let $B \geq 0$. Then, the difference scheme (2.1) is stable with respect to its right-hand side and initial data, and its solution satisfies the a priori estimate*

$$\|y_{n+1}\|_A \leq m_\varepsilon \left(\|y_0\|_A + \|y_{i,0}\|_D + \sum_{s=1}^n \tau \|\varphi_s\|_{D^{-1}} \right), \quad m_\varepsilon = \sqrt{\frac{1 + \varepsilon}{\varepsilon}}. \tag{2.4}$$

For simplicity, we restrict our analysis to three-level operator-difference schemes with weights:

$$y_{i\bar{t}} + A y^{(\sigma_1, \sigma_2)} = \varphi, \quad y_0 = u_0, \quad y_1 = u_1. \tag{2.5}$$

Here, $y^{(\sigma_1, \sigma_2)} = \sigma_1 \hat{y} + (1 - \sigma_1 - \sigma_2)y + \sigma_2 \check{y}$. Note that the identity

$$y^{(\sigma_1, \sigma_2)} = y + \tau(\sigma_1 - \sigma_2)y_i^0 + 0.5\tau^2(\sigma_1 + \sigma_2)y_{i\bar{t}} \tag{2.6}$$

can be used to transform scheme (2.5) into the canonical form (2.1).

Denote by \tilde{y} the solution to problem (2.5) in which the right-hand side, the initial condition, and the operator are perturbed:

$$\tilde{y}_{i\bar{t}} + \tilde{A} \tilde{y}^{(\sigma_1, \sigma_2)} = \tilde{\varphi}, \quad \tilde{y}_0 = \tilde{u}_0, \quad \tilde{y}_1 = \tilde{u}_1. \tag{2.7}$$

From (2.5) and (2.7), we obtain a problem for the solution's perturbation $z_n = \tilde{y}_n - y_n$:

$$z_{i\bar{t}} + A z^{(\sigma_1, \sigma_2)} = (\tilde{\varphi} - \varphi) - (\tilde{A} - A) \tilde{y}^{(\sigma_1, \sigma_2)}, \tag{2.8}$$

$$z_0 = \tilde{u}_0 - u_0, \quad z_1 = \tilde{u}_1 - u_1. \tag{2.9}$$

Substituting (2.6) into Eq. (2.8), we transform this equation into the canonical form

$$D z_{i\bar{t}} + B z_i^0 + A z = \psi, \quad z_0 = \tilde{u}_0 + u_0, \quad z_1 = \tilde{u}_1 - u_1, \tag{2.10}$$

where

$$B = \tau(\sigma_1 - \sigma_2)A, \quad D = E + 0.5\tau^2(\sigma_1 + \sigma_2)A, \quad \psi = (\tilde{\varphi} - \varphi) - (\tilde{A} - A) \tilde{y}^{(\sigma_1, \sigma_2)}.$$

In what follows, we assume that the operators A and \tilde{A} in (2.5) and (2.7) are self-adjoint, constant, and positive:

$$A^* = A > 0, \quad \tilde{A}^* = \tilde{A} > 0. \tag{2.11}$$

To define a measure of perturbation in A (which is generally unbounded), we introduce a constant α appearing in the inequality

$$|((\tilde{A} - A)y, v)| \leq \alpha \|y\|_A \|v\|_{\tilde{A}}, \quad y, v \in H, \quad \alpha \geq 0. \tag{2.12}$$

Then, the strong unconditional stability of the solution to problem (2.5) means that there exist positive constants M_1, M_2 , and M_3 that are independent of n or the initial data in problems (2.5), (2.7) and are such that, for a sufficiently small $\tau \leq \tau_0$,

$$\|\tilde{y} - y\|_{(1)} \leq M_1 \|\tilde{y}_0 - y_0\|_{(1)} + M_2 \|\tilde{\varphi} - \varphi\|_{(2)} + \alpha M_3 (\|\tilde{y}_0\|_{(3)} + \|\tilde{\varphi}\|_{(4)}) \tag{2.13}$$

for any $y_0, \tilde{y}_0; \varphi, \tilde{\varphi} \in H$; here, $\|\cdot\|_{(k)}$ are norms in H . If (2.13) holds for an arbitrary H , then the scheme is called absolutely stable.

3. STRONG STABILITY IN H

The stability analysis of three-level schemes in the H norm with respect to their initial data and right-hand sides is a difficult task. In [1, 10], the stability analysis is based on the superposition principle, which makes it possible to discard the unnatural condition of ρ -stability. These ideas are used below to obtain a priori estimates of strong stability.

Theorem 1. *Suppose that the operators A and \tilde{A} satisfy (2.11) and (2.12), and the weights σ_1 and σ_2 satisfy the conditions*

$$\begin{aligned} \sigma_1 \geq \sigma_2 \geq 0, \quad \sigma_1 + \sigma_2 \leq 1, \\ \sigma_1 + \sigma_2 \geq (1 + \varepsilon)/2 - 2/(\tau^2 c), \quad c = \max\{\|A\|, \|\tilde{A}\|\}. \end{aligned} \tag{3.1}$$

Then, $z = \tilde{y} - y$ [the difference of the solutions to (2.7) and (2.5)] satisfies the following strong-stability estimate in the L_2 norm:

$$\|z_{n+1}\| \leq m_\varepsilon \left[\|z_0\| + \|z_{t,0}\|_{D_1} + \sum_{s=1}^n \tau \|\tilde{\varphi}_s - \varphi_s\|_{A^{-1}} + \alpha t_n \left(\|\tilde{y}_0\|_{\tilde{A}} + \|\tilde{y}_{t,0}\|_{\tilde{D}} + \sum_{s=1}^n \tau \|\tilde{\varphi}_s\|_{\tilde{D}^{-1}} \right) \right], \tag{3.2}$$

where $D_1 = A^{-1} + 0.5\tau^2(\sigma_1 + \sigma_2)E$ and $\tilde{D} = E + 0.5\tau^2(\sigma_1 + \sigma_2)\tilde{A}$.

Proof. Premultiplying Eq. (2.10) by A^{-1} , we obtain another canonical form of the difference scheme for z :

$$D_1 z_{it} + B_1 z_i^* + A_1 z = \psi_1, \quad z_0 = \tilde{u}_0 - u_0, \quad z_1 = \tilde{u}_1 - u_1.$$

Here, $A_1 = E$, $B_1 = \tau(\sigma_1 - \sigma_2)E \geq 0$, and $\psi_1 = A^{-1}(\tilde{\varphi} - \varphi) - A^{-1}(\tilde{A} - A)\tilde{y}^{(\sigma_1, \sigma_2)}$.

Let $z = \overset{1}{z} + \overset{2}{z}$, where $\overset{1}{z}$ is a solution to the inhomogeneous equation (2.7) with homogeneous initial data,

$$D_1 \overset{1}{z}_{it} + B_1 \overset{1}{z}_i^* + A_1 \overset{1}{z} = \psi_1, \quad \overset{1}{z}_0 = 0, \quad \overset{1}{z}_1 = 0, \tag{3.3}$$

and $\overset{2}{z}$ is a solution to the homogeneous equation (2.7) with inhomogeneous initial data:

$$D_1 \overset{2}{z}_{it} + B_1 \overset{2}{z}_i^* + A_1 \overset{2}{z} = \psi_1, \quad \overset{2}{z}_0 = \tilde{u}_0 - u_0, \quad \overset{2}{z}_1 = \tilde{u}_1 - u_1. \tag{3.4}$$

Following [2, p. 124], a solution to problem (3.3) is sought in the form of the sum (superposition principle):

$$\overset{1}{z}_n = \sum_{s=1}^n \tau g_{ns}, \quad n = 1, 2, \dots, \quad z_0 = 0,$$

where g_{ns} is a function of n for each fixed $s = 1, 2, \dots, n - 1$ and satisfies the equation

$$D_1(g_{ns})_{it} + B_1(g_{ns})_i^* + A_1 g_{ns} = 0 \tag{3.5}$$

with the initial conditions

$$g_{ss} = 0, \quad (0.5\tau B_1 + D_1) \frac{g_{s+1,s} - g_{ss}}{\tau} = \psi_{1s}. \tag{3.6}$$

Since $A^{-1} \geq E/\|A\|$ for self-adjoint operators, we conclude that, under the conditions in the theorem,

$$D_1 - 0.25(1 + \varepsilon)\tau^2 A_1 = A^{-1} + \frac{\tau^2}{2} \left(\sigma_1 + \sigma_2 - \frac{1 + \varepsilon}{2} \right) E \geq 0.$$

Since $D_1 > 0$ and H is a finite-dimensional space, $D_1 \geq \delta E$. For $B_1 \geq 0$ and $w = (g_{s+1,s} - g_{ss})/\tau$, the scalar product of $(0.5\tau B_1 + D_1)w = \psi_1$ and w yields

$$\|w\|_{D_1}^2 \leq ((0.5\tau B_1 + D_1)w, w) = (A^{-1}(\tilde{\varphi} - \varphi), w) + (A^{-1}(\tilde{A} - A)\tilde{y}^{(\sigma_1, \sigma_2)}, w). \tag{3.7}$$

To estimate the right-hand side of (3.7), we use the Cauchy-Schwarz inequality and property (2.12) to

obtain

$$(A^{-1}(\tilde{\varphi} - \varphi), w) \leq \|\tilde{\varphi} - \varphi\|_{A^{-1}} \|w\|_{A^{-1}},$$

$$(A^{-1}(\tilde{A} - A)\tilde{y}^{(\sigma_1, \sigma_2)}, w) = ((\tilde{A} - A)\tilde{y}^{(\sigma_1, \sigma_2)}, A^{-1}w) \leq \alpha \|\tilde{y}^{(\sigma_1, \sigma_2)}\|_{\tilde{A}} \|w\|_{A^{-1}}.$$

Substituting these expressions into (3.7) and using $\|w\|_{A^{-1}} \leq \|w\|_{D_1}$, we obtain the estimate

$$\|(g_t)_{ss}\|_{D_1} \leq \|\tilde{\varphi}_s - \varphi_s\|_{A^{-1}} + \alpha \|\tilde{y}_s^{(\sigma_1, \sigma_2)}\|_{\tilde{A}}. \tag{3.8}$$

Since conditions (2.3) are satisfied by D_1, B_1 , and A_1 , the application of Lemmas 1 and 2 to the solution to problem (3.5), (3.6) yields

$$\|g_{n+1, s}\| \leq m_\varepsilon \|(g_t)_{ss}\|_{D_1}.$$

Summing this relation over s from 1 to n and taking into account (3.8) and the inequality

$$\|z_{n+1}\| \leq \sum_{s=1}^n \tau \|g_{n+1, s}\|,$$

we find the estimate

$$\|z_{n+1}\| \leq m_\varepsilon \left(\sum_{s=1}^n \tau \|\tilde{\varphi}_s - \varphi_s\|_{A^{-1}} + \alpha \sum_{s=1}^n \tau \|\tilde{y}_s^{(\sigma_1, \sigma_2)}\|_{\tilde{A}} \right). \tag{3.9}$$

Applying Lemmas 1 and 2 to the solution to problem (3.4) produces

$$\|z_{n+1}\|^2 \leq m_\varepsilon (\|z_0\| + \|z_{t0}\|_{D_1}). \tag{3.10}$$

To derive the final estimate (3.2), we have to estimate the sum

$$\sum_{s=1}^n \tau \|\tilde{y}_s^{(\sigma_1, \sigma_2)}\|_{\tilde{A}}.$$

To this end, we write problem (2.7) for a perturbed solution in the canonical form

$$\tilde{D}\tilde{y}_{it} + \tilde{B}\tilde{y}_t + \tilde{A}\tilde{y} = \tilde{\varphi}, \quad \tilde{y}_0 = \tilde{u}_0, \quad \tilde{y}_1 = \tilde{u}_1,$$

$$\tilde{D} = E + 0.5\tau^2(\sigma_1 + \sigma_2)\tilde{A}, \quad \tilde{B} = \tau(\sigma_1 - \sigma_2)\tilde{A}, \quad \tilde{A}^* = \tilde{A} > 0,$$

and apply Lemma 3:

$$\sum_{s=1}^n \tau \|\tilde{y}_s^{(\sigma_1, \sigma_2)}\|_{\tilde{A}} \leq t_n \max_{0 \leq k \leq n+1} \|\tilde{y}_k\|_{\tilde{A}} \leq t_n m_\varepsilon \left(\|\tilde{y}_0\|_{\tilde{A}} + \|\tilde{y}_{t0}\|_{\tilde{D}} + \sum_{s=1}^n \tau \|\tilde{\varphi}_s\|_{\tilde{D}^{-1}} \right).$$

Substituting these estimates into the triangle inequality $\|z\| \leq \|z^1\| + \|z^2\|$, we arrive at the desired estimate, which completes the proof of the theorem.

Example 1. In the rectangle

$$\bar{Q}_T = \bar{\Omega} \times [0, T], \quad \bar{\Omega} = \{x : 0 \leq x \leq l\}, \quad 0 \leq t \leq T,$$

consider the Dirichlet problem for the one-dimensional hyperbolic equation

$$\frac{\partial^2 u}{\partial t^2} = Lu + f(x, t), \quad Lu = \frac{\partial}{\partial x} \left(k(x) \frac{\partial u}{\partial x} \right), \quad 0 < x < l, \quad 0 < t \leq T, \tag{3.11}$$

$$u(0, t) = u(l, t) = 0, \quad 0 < t \leq T, \quad u(x, 0) = u_0(x), \quad \frac{\partial u}{\partial t}(x, 0) = u_1(x), \quad 0 \leq x \leq l, \tag{3.12}$$

$$0 < c_1 \leq k(x) \leq c_2, \quad x \in \bar{\Omega}, \quad c_1, c_2 = \text{const.} \tag{3.13}$$

Since we seek a continuous solution, the following matching conditions are assumed to hold for it:

$$u_0(x) = u_1(x) = 0, \quad x \in \Gamma, \quad f(x, t) = 0, \quad x \in \Gamma_T, \tag{3.14}$$

where Γ is the boundary of Ω , and Γ_T is the lateral surface of Q_T . On a uniform grid $\bar{\omega} = \bar{\omega}_h \times \bar{\omega}_\tau$, $\bar{\omega}_h = \{x_i = ih, i = 0, 1, \dots, N, hN = l\}$, where $\bar{\omega}_\tau = \{t_n = n\tau, n = 0, 1, \dots, N_0, \tau N_0 = T\}$, and the steps h and τ are constant, the differential problem (3.11)–(3.13) is approximated by an ordinary difference scheme with weights:

$$y_{i\bar{i},i} = (ay_{\bar{x}}^{(\sigma_1, \sigma_2)})_{x,i} + \varphi_i, \quad i = 1, 2, \dots, N-1, \tag{3.15}$$

$$y_0^{n+1} = y_N^{n+1} = 0, \quad n = 0, 1, \dots, N_0, \quad (y^0)_i = u_{0i}, \quad (y^0)_i = \bar{u}_{1i}. \tag{3.16}$$

Here, $y^{(\sigma_1, \sigma_2)} = \sigma_1 \hat{y} + (1 - \sigma_1 - \sigma_2) + \sigma_2 \check{y}$, $\bar{u}_1(x) = u_1(x) + 0.5\tau[Lu_0 + f(x, 0)]$, and $a = a(x)$, $x \in \omega_h^+$ is a standard template functional satisfying the condition $0 < c_1 \leq a(x) \leq c_2$ at $x \in \omega_h^+ = \omega_h \cup \{x_N = l\}$. The difference scheme (3.15), (3.16) is reduced to the operator scheme

$$y_{i\bar{i}} + Ay^{(\sigma_1, \sigma_2)} = \varphi, \quad y_0 = u_0, \quad y_{i0} = \bar{u}_1$$

by setting $y = (y_1, \dots, y_{N-1})^T$, $\varphi = (\varphi_1, \dots, \varphi_{N-1})^T$, $\varphi_i = f(x_i)$, and $(Ay)_i = -(ay_{\bar{x}})_{x,i}$, $i = 1, 2, \dots, N-1$, $y_0 = y_N = 0$. Here, A is an operator from H into H , where $H = \mathcal{H}_h$ is the linear space of vectors of the form $(v(x_1), \dots, v(x_{N-1}))^T$, with the inner product and the norm defined, as usual, by

$$(y, v) = \sum_{i=1}^{N-1} h y_i v_i, \quad \|v\| = \sqrt{(v, v)}.$$

Consider the perturbed problem

$$\begin{aligned} \tilde{y}_{i\bar{i},i} &= (\tilde{a}\tilde{y}_{\bar{x}}^{(\sigma_1, \sigma_2)})_{x,i} + \tilde{\varphi}_i, \quad i = 1, 2, \dots, N-1, \\ \tilde{y}_0^{n+1} &= \tilde{y}_N^{n+1} = 0, \quad n = 0, 1, \dots, N_0, \quad (\tilde{y}^0)_i = \tilde{u}_{0i}, \quad (\tilde{y}^0)_i = \tilde{\tilde{u}}_{1i}, \end{aligned}$$

where $0 < \tilde{c}_1 \leq \tilde{a}(x) \leq \tilde{c}_2$, $x \in \omega_h^+$, and

$$(\tilde{A}\tilde{y})_i = -(\tilde{a}\tilde{y}_{\bar{x}})_{x,i}, \quad i = 1, 2, \dots, N-1, \quad \tilde{y}_0 = \tilde{y}_N = 0.$$

The properties of the operator A have been studied extensively (see [1]). In particular, inequalities (2.11) hold for it. Before applying Theorem 1, we show that the operator

$$(\tilde{A} - A)y = -((\tilde{a} - a)y_{\bar{x}})_x$$

satisfies condition (2.12) with a constant $\alpha = c_0 \|\tilde{a} - a\|_C$, where $c_0 = (c_1 \tilde{c}_1)^{-1/2}$ and $\|\cdot\|_C = \max_{x \in \omega_h} |\cdot|$. Specifically,

$$|((\tilde{A} - A)y, v)| \leq \alpha \|y\|_A \|v\|_{\tilde{A}}, \quad y, v \in H, \quad \alpha \geq 0.$$

Indeed, since $|\tilde{a} - a| \leq \alpha \tilde{a}^{1/2} a^{1/2}$ in this case, we use the first Green difference formula to obtain

$$|((\tilde{A} - A)y, v)| = |((\tilde{a} - a)y_{\bar{x}}, v_{\bar{x}})| \leq \alpha \|\tilde{a}^{1/2} y_{\bar{x}}\| \|a^{1/2} v_{\bar{x}}\| = \alpha \|y\|_A \|v\|_A,$$

where

$$(y, v) = \sum_{i=1}^N h y_i v_i.$$

Next, applying Theorem 1 to the solution to the difference scheme (3.15), (3.16), we obtain the a priori estimate (3.2), which means that the scheme is strongly stable in the L_2 norm under the following conditions on σ_1 and σ_2 :

$$\sigma_1 \geq \sigma_2 \geq 0, \quad \frac{1 + \varepsilon}{2} - \frac{2}{\tau^2 c} \leq \sigma_1 + \sigma_2 \leq 1, \quad c = \max\{\|A\|, \|\tilde{A}\|\} < \frac{4c_3}{h^2}, \quad c_3 = \max\{c_2, \tilde{c}_2\}.$$

4. STABILITY IN THE H_A NORM

To derive strong-stability estimates in stronger norms, we use another measure of perturbation in A (see [7–9]):

$$\|(\tilde{A} - A)v\| \leq \alpha \|\tilde{A}v\|, \quad \alpha \geq 0, \quad v \in H. \tag{4.1}$$

We will need to use the following lemmas.

Lemma 4. *Let the operator A in the two-level operator-difference scheme*

$$By_t + Ay = \varphi, \quad t \in \omega_\tau, \quad y(0) = u_0 \tag{4.2}$$

be self-adjoint, constant, and positive:

$$A = A^* > 0. \tag{4.3}$$

Then, under the condition

$$B \geq 0.5\tau A, \tag{4.4}$$

the difference scheme (4.2) is stable with respect to its right-hand side and initial data, and the solution to the problem satisfies the a priori estimate

$$\|y_{n+1}\|_A \leq \|y_0\|_A + \|\varphi_0\|_{A^{-1}} + \|\varphi_n\|_{A^{-1}} + \sum_{k=1}^n \tau \|\varphi_{\bar{i},k}\|_{A^{-1}}. \tag{4.5}$$

Proof. This theorem is usually proved by the method of stationary inhomogeneities [2, p. 148]. We present another, very simple proof of this result. By using the identity $v = v^{(0.5)} - 0.5\tau vt$, scheme (4.2) can be rewritten as

$$(B - 0.5\tau A)y_t + Ay^{(0.5)} = \varphi,$$

where $y^{(\sigma)} = \sigma \hat{y} + (1 - \sigma)y$. Multiplying this equation by $2\tau y_t$, and taking into account conditions (4.3) and (4.4), we obtain the energy inequality

$$\|y_{n+1}\|_A^2 - 2(y_{n+1}, \varphi_n) + \|\varphi_n\|_{A^{-1}}^2 \leq \|y_n\|_A^2 - 2(y_n, \varphi_n) + \|\varphi_n\|_{A^{-1}}^2. \tag{4.6}$$

By using $\|A^{1/2}v\| = \|v\|_A$ and $\|A^{-1/2}v\| = \|v\|_{A^{-1}}$, (4.6) is written as

$$\|y_{n+1} - A^{-1}\varphi_n\|_A \leq \|y_n - A^{-1}\varphi_n\|_A. \tag{4.7}$$

Next, we add $A^{-1}\varphi_{n-1}$ to and subtract it from the right-hand side of (4.7) to obtain the recurrence relation

$$\|y_{k+1} - A^{-1}\varphi_k\|_A \leq \|y_k - A^{-1}\varphi_{k-1}\|_A + \tau \|\varphi_{\bar{i},k}\|_{A^{-1}}.$$

Summing over all $k = 1, 2, \dots, n$, with the use of $\|u\| - \|v\| \leq \|u - v\|$ in the left-hand side, produces

$$\|y_{n+1}\|_A \leq \|\varphi_n\|_{A^{-1}} + \|y_1 - A^{-1}\varphi_0\|_A + \sum_{k=1}^n \tau \|\varphi_{\bar{i},k}\|_{A^{-1}}. \tag{4.8}$$

The term $\|y_1 - A^{-1}\varphi_0\|_A$ is estimated by applying (4.7), with $n = 0$:

$$\|y_1 - A^{-1}\varphi_0\|_A \leq \|y_0 - A^{-1}\varphi_0\|_A.$$

Substituting this into (4.8) and using the triangle inequality, we arrive at the desired estimate (4.5). The proof of Lemma 4 is completed.

Lemma 5. *Let the conditions in Lemma 3 be fulfilled. Then, the difference scheme (2.1) is stable with respect to its right-hand side and initial data, and its solution satisfies the estimate*

$$\|Ay_{n+1}\| \leq m_\epsilon \left(\|Ay_0\| + \|y_{i,0}\|_{\bar{D}} + \|\varphi_1\| + \|\varphi_n\| + \sum_{k=2}^n \tau \|\varphi_{i,k}\| \right), \tag{4.9}$$

$$\bar{D} = AD.$$

Proof. We premultiply (2.1) by the operator A to reduce the original problem to a convenient canonical form, i.e., to the three-level operator-difference scheme

$$\bar{D}y_{it} + \bar{B}y_t + \bar{A}y = \bar{\varphi}, \quad y_0 = u_0, \quad y_1 = u_1. \tag{4.10}$$

Here,

$$\bar{D} = AD, \quad \bar{B} = AB, \quad \bar{A} = A^2, \quad \bar{\varphi} = A\varphi.$$

Defining \mathcal{A} and \mathcal{B} as the operator matrices

$$\mathcal{A} = \begin{pmatrix} \bar{A} & 0 \\ 0 & \frac{1}{\tau^2}\bar{R} \end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} \bar{B} + 0.5\tau\bar{A} & \frac{1}{\tau}\bar{R} \\ -\frac{1}{\tau}\bar{R} & \frac{1}{2\tau}\bar{R} \end{pmatrix}, \quad \bar{R} = \bar{D} - \frac{\tau^2}{4}\bar{A}, \tag{4.11}$$

we reduce (4.10) to the two-level scheme

$$\mathcal{B}Y_t + \mathcal{A}Y = \Phi, \quad Y_t \in H^2, \tag{4.12}$$

for $Y_n = \{0.5(y_n + y_{n-1}), y_n - y_{n-1}\}$ and $\Phi_n = \{\varphi_n, 0\}$.

Applying Lemma 4, we obtain a stability estimate for the solution to scheme (4.12) with respect to its right-hand side and initial data:

$$\|Y_{n+1}\|_{\mathcal{A}} \leq \|Y_1\|_{\mathcal{A}} + \|A\varphi_1\|_{\bar{A}^{-1}} + \|A\varphi_n\|_{\bar{A}^{-1}} + \sum_{k=1}^n \tau \|A\varphi_{i,k}\|_{\bar{A}^{-1}}.$$

Substituting the estimates from Lemma 2 into this inequality and using (4.11), we find the following a priori estimate for the solution to Eq. (4.10):

$$\|Ay_{n+1}\| \leq \sqrt{\frac{1+\epsilon}{\epsilon}} \left(\|Ay_0\| + \|y_{i,0}\|_{\bar{D}} + \|\varphi_1\| + \|\varphi_n\| + \sum_{k=2}^n \tau \|\varphi_{i,k}\| \right).$$

The proof of Lemma 5 is completed.

The following theorem states the strong stability of the scheme with weights (2.5).

Theorem 2. *Let $A^* = A > 0$ and $\tilde{A}^* = \tilde{A} > 0$ be constant and satisfy condition (4.1), and let inequalities (3.1) be valid. Then, the solution $z = \tilde{y} - y$ to problem (2.8), (2.9) satisfies the a priori estimate*

$$\begin{aligned} \|z_{n+1}\|_A &\leq m_\epsilon \left(\|z_0\|_A + \|z_{i,0}\|_D + \sum_{k=1}^n \tau \|\tilde{\varphi}_k - \varphi_k\| \right. \\ &\left. + \alpha t_n \left(\|\tilde{A}\tilde{y}_0\| + \|\tilde{y}_{i,0}\|_{\bar{D}} + \|\tilde{\varphi}_1\| + \max_{0 \leq k \leq n} \|\tilde{\varphi}_k\| + \sum_{k=2}^n \tau \|\tilde{\varphi}_{i,k}\| \right) \right). \end{aligned} \tag{4.13}$$

Proof. Consider the canonical form of the problem for z (see (2.8), (2.9)). Since $D^{-1} < E$, we conclude,

by Lemma 3, that

$$\|z_{n+1}\|_A \leq m_\varepsilon \left(\|z_0\|_A + \|z_{i,0}\|_D + \sum_{k=1}^n \tau \|\Psi_k\| \right), \tag{4.14}$$

where $\Psi_k = (\tilde{\varphi}_k - \varphi_k) - (\tilde{A} - A)\tilde{y}_k^{(\sigma_1, \sigma_2)}$.

To estimate $\|\Psi_k\|$, we use the triangle inequality and property (4.1) the operator perturbation:

$$\|\Psi_k\| \leq \|\tilde{\varphi}_k - \varphi_k\| + \alpha \|\tilde{A}\tilde{y}_k^{(\sigma_1, \sigma_2)}\|.$$

The sum

$$\sum_{k=1}^n \tau \|\tilde{A}\tilde{y}_k^{(\sigma_1, \sigma_2)}\|$$

is estimated by applying Lemma 5:

$$\sum_{k=1}^n \tau \|\tilde{A}\tilde{y}_k^{(\sigma_1, \sigma_2)}\| \leq t_n m_\varepsilon \left(\|\tilde{A}\tilde{y}_0\| + \|\tilde{y}_{i,0}\|_{\tilde{D}} + \|\tilde{\varphi}_1\| + \max_{0 \leq k \leq n} \|\tilde{\varphi}_k\| + \sum_{k=2}^n \tau \|\tilde{\varphi}_{\bar{i},k}\| \right).$$

Substituting the last inequalities into (4.14), we arrive at (4.13), which completes the proof of Theorem 2.

Example 2. Let us illustrate the above results by using a two-dimensional initial-boundary value problem for the wave equation.

In the parallelepiped

$$\bar{\Omega} \times [0, T], \quad \bar{\Omega} = \{x \mid x = (x_1, x_2), 0 \leq x_\alpha \leq l_\alpha, \alpha = 1, 2\}$$

consider the equation

$$\frac{\partial^2 u}{\partial t^2} = Lu + f(x, t), \quad Lu = \frac{\partial}{\partial x_1} \left(k_1(x) \frac{\partial u}{\partial x_1} \right) + \frac{\partial}{\partial x_2} \left(k_2(x) \frac{\partial u}{\partial x_2} \right), \quad x \in \Omega, \quad 0 < t \leq T, \tag{4.15}$$

$$u(x, t) = 0, \quad x \in \partial\Omega, \quad 0 < t \leq T, \tag{4.16}$$

$$u(x, 0) = u_0(x), \quad x \in \Omega, \quad \frac{\partial u}{\partial t}(x, 0) = u_1(x), \quad x \in \Omega, \tag{4.17}$$

under the assumption that $0 < c_1 \leq k_\alpha(x) \leq c_2, x \in \Omega$.

We introduce a uniform grid $\bar{\omega} = \bar{\omega}_h \times \bar{\omega}_\tau$, where $\bar{\omega}_\tau = \{t_n = n\tau, n = 0, 1, \dots, N_0; \tau N_0 = T\}$, $\bar{\omega}_h = \bar{\omega}_1 \times \bar{\omega}_2 = \omega_h \cup \gamma_h$, γ_h is the set of boundary nodes, $\bar{\omega}_\alpha = \{x_\alpha = i_\alpha h_\alpha, i_\alpha = 0, 1, \dots, N_\alpha, N_\alpha h_\alpha = l_\alpha\}$, and $\omega_\alpha^+ = \omega_\alpha \cup \{x_{N_\alpha} = l_\alpha\}$.

Let $H = \mathcal{H}_h$ be the set $\{y(x)\}_{x \in \bar{\omega}_h}$ of grid functions defined on $\bar{\omega}_h$ and satisfying the condition $y(x) = 0, x \in \gamma_h$. The inner product and the norms are defined, as usual, by the relations

$$(u, v) = \sum_{x \in \omega_h} u(x)v(x)h_1h_2, \quad \|v\| = \sqrt{(v, v)}, \quad \|v_{\bar{x}_\alpha}\|^2 = \sum_{i_\alpha=1}^{N_\alpha} \sum_{i_\beta=1}^{N_\beta-1} v_{\bar{x}_\alpha}^2 h_1 h_2,$$

$$\|v\|_1^2 = \|v_{\bar{x}_1}\|^2 + \|v_{\bar{x}_2}\|^2, \quad \|v\|_2 = \|v_{\bar{x}_1 x_1} + v_{\bar{x}_2 x_2}\|, \quad \|v\|_C = \max_{x \in \omega_h} |v(x)|.$$

On the grid $\bar{\omega}$, the differential problem (4.15)–(4.17) is approximated by the difference scheme

$$y_{i,t} = (a_1(x)y_{\bar{x}_1}^{(\sigma_1, \sigma_2)})_{x_1} + (a_2(x)y_{\bar{x}_2}^{(\sigma_1, \sigma_2)})_{x_2} + \varphi(x, t), \quad x \in \omega_h, \quad t \in \omega_\tau, \tag{4.18}$$

$$y(x, t) = 0, \quad x \in \gamma_h, \quad t \in \omega_\tau, \tag{4.19}$$

$$y(x, 0) = u_0(x), \quad x \in \omega_h, \quad y_t(x, 0) = \bar{u}_1(x), \quad x \in \omega_h, \tag{4.20}$$

where a_1, a_2 , and φ are template functionals, and $0 < c_1 \leq a_\alpha(x) \leq c_2$ for $x \in \omega_1^+ \times \omega_2^+$.

The difference scheme (4.18)–(4.20) is reduced to the operator scheme (2.5) by setting

$$(Av)(x) = -(a_1(x)v_{\bar{x}_1})_{x_1} - (a_2(x)v_{\bar{x}_2})_{x_2}, \quad x \in \omega_h, \quad v(x) = 0, \quad x \in \gamma_h.$$

Consider the corresponding perturbed problem

$$\tilde{y}_{it} = (\tilde{a}_1(x)\tilde{y}_{\bar{x}_1}^{(\sigma_1, \sigma_2)})_{x_1} + (\tilde{a}_2(x)\tilde{y}_{\bar{x}_2}^{(\sigma_1, \sigma_2)})_{x_2} + \tilde{\varphi}(x, t), \quad x \in \omega_h, \quad t \in \omega_\tau, \tag{4.21}$$

$$\tilde{y}(x, t) = 0, \quad x \in \gamma_h, \quad t \in \omega_\tau, \tag{4.22}$$

$$\tilde{y}(x, 0) = \tilde{u}_0(x), \quad x \in \omega_h, \quad \tilde{y}_t(x, 0) = \tilde{u}_1(x), \quad x \in \omega_h, \tag{4.23}$$

where

$$0 < \tilde{c}_1 \leq \tilde{a}_\alpha \leq \tilde{c}_2, \quad |(\tilde{a}_\alpha^{-1})_{\bar{x}_\alpha}| \leq \tilde{c}_3, \quad \alpha = 1, 2. \tag{4.24}$$

Defining the operator

$$(\tilde{A}\tilde{v})(x) = -(\tilde{a}_1(x)\tilde{v}_{\bar{x}_1})_{x_1} - (\tilde{a}_2(x)\tilde{v}_{\bar{x}_2})_{x_2}, \quad x \in \omega_h, \quad \tilde{v}(x) = 0, \quad x \in \gamma_h,$$

we reduce scheme (4.21)–(4.23) to the operator form (2.7). The operators A and \tilde{A} are self-adjoint and positive [1].

Let us show that condition (4.1) is fulfilled for the operator

$$(\tilde{A} - A)y = -((\tilde{a}_1 - a_1)y_{\bar{x}_1})_{x_1} - ((\tilde{a}_2 - a_2)y_{\bar{x}_2})_{x_2}, \tag{4.25}$$

i.e., $\|(\tilde{A} - A)y\| \leq \alpha \|\tilde{A}y\|$ with a nonnegative constant

$$\alpha = c_4(\alpha_1 + \alpha_2), \quad c_4 = \text{const} > 0, \tag{4.26}$$

$$\alpha_1 = \frac{1}{\tilde{c}_1} \max\{\|\tilde{a}_1 - a_1\|_C, \|\tilde{a}_2 - a_2\|_C\}, \quad \alpha_2 = \max\{\|\tilde{a}_{1x_1} - a_{1x_1}\|_C, \|\tilde{a}_{2x_2} - a_{2x_2}\|_C\}. \tag{4.27}$$

To prove this assertion, we use the following lemma (see [2, p. 184]).

Lemma 6. For any grid function $y \in \mathcal{H}_h$,

$$\|y\|_1 \leq M \|Ay\|, \quad M = \text{const} > 0. \tag{4.28}$$

Applying the identity

$$\begin{aligned} ((\tilde{a} - a)y_{\bar{x}})_x &= \left(\left(\frac{\tilde{a} - a}{\tilde{a}} \right) \tilde{a}y_{\bar{x}} \right)_x \\ &= \left(\frac{\tilde{a} - a}{\tilde{a}} \right)_+ (\tilde{a}y_{\bar{x}})_x + (\tilde{a}(\tilde{a}^{-1})_x(\tilde{a} - a)_+ + (\tilde{a}_x - a_x))y_{\bar{x}}, \quad v_+ = v(x+h) \end{aligned}$$

to operator (4.25) and using condition (4.27), we obtain

$$\|(\tilde{A} - A)y\| \leq \alpha_1 (\|(\tilde{a}_1 y_{\bar{x}_1})_{x_1}\| + \|(\tilde{a}_2 y_{\bar{x}_2})_{x_2}\|) + (\tilde{c}\alpha_1 + \alpha_2)(\|y_{\bar{x}_1}\| + \|y_{\bar{x}_2}\|), \quad \tilde{c} = \tilde{c}_1 \tilde{c}_2 \tilde{c}_3. \tag{4.29}$$

Applying the algebraic inequality $a + b \leq \sqrt{2(a^2 + b^2)}$ gives the estimate

$$\|(\tilde{a}_1 \tilde{y}_{\bar{x}_1})_{x_1}\| + \|(\tilde{a}_2 y_{\bar{x}_2})_{x_2}\| \leq \sqrt{2} \|\tilde{A}y\|, \quad \|y_{\bar{x}_1}\| + \|y_{\bar{x}_2}\| \leq \sqrt{2} \|y\|_1. \tag{4.30}$$

Substituting (4.30) into (4.29) and using Lemma 6, we arrive at (4.1) with a constant $\alpha = c_4(\alpha_1 + \alpha_2)$, where $c_4 = \sqrt{2} \max\{M, 1 + \tilde{c}M\}$. Consequently, by Theorem 2, we conclude that the solution to the difference scheme (4.18)–(4.20) satisfies the a priori estimate (4.13), which means that the scheme is strongly stable.

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