OLD DOMINION UNIVERSITY

Ordinary Differential Equations II

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1. Second-order ODEs
2. Particle dynamics
3. Applications:
   a) Oscillatory motion and chaos, b) Projectile motion,
   c) Classical scattering, d) Planetary and satellite motion

updated 4 March 2022

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Part 1:

Second-order ODEs

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Higher-order ODEs

- In the first part we considered solutions of first-order ordinary differential equations by finite difference methods.
- Many problems in physics are governed by higher-order ODEs. The second-order ODEs are most common ODEs.
- In general, a higher-order ODE can be replaced by a system of first-order ODEs.

Example: Newton's second law provides us with equation of motion

$$\frac{d^2x}{dt^2} = f\left(t, x, \frac{dx}{dt}\right)$$

Introducing $dx/dt = v$, we get a system of two first-order ODEs

$$\frac{dx}{dt} = v, \qquad \frac{dv}{dt} = f(t, x, v)$$

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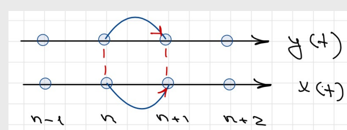
A system of first-order ODEs

A system of first-order ODEs can be solved by any of the methods developed for solving single ODEs.

- Care must be taking to ensure the proper copying all the solutions.
- When predictor – corrector or Runge-Kutta methods are used, each step must be applied to all the equations before proceeding to the next step.
- The step-size must be the same for all of the equations.



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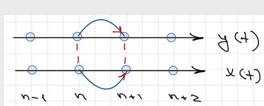
A system of two first-order ODEs

$$\frac{dx}{dt} = f_1(t, x, y)$$

$$\frac{dy}{dt} = f_2(t, x, y)$$



Explicit Euler method

$$x_{n+1} = x_n + f_1(t_n, x_n, y_n)\Delta t$$

$$y_{n+1} = y_n + f_2(t_n, x_n, y_n)\Delta t$$

Predictor-corrector

$$x_{n+1}^P = x_n + f_1(t_n, x_n, y_n)\Delta t$$

$$y_{n+1}^P = y_n + f_2(t_n, x_n, y_n)\Delta t$$

$$x_{n+1}^C = x_n + \frac{1}{2}[f_1(t_n, x_n, y_n) + f_1(t_{n+1}, x_{n+1}^P, y_{n+1}^P)]\Delta t$$

$$y_{n+1}^C = y_n + \frac{1}{2}[f_2(t_n, x_n, y_n) + f_2(t_{n+1}, x_{n+1}^P, y_{n+1}^P)]\Delta t$$

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Example: C++ 4th order RK for a system of two eqs.

```
double rk4_2nd(double(*d1x)(double, double, double),
               double(*d1y)(double, double, double),
               double ti, double xi, double yi, double tf,
               double& xf, double& yf)
{
      double h,t,k1x,k2x,k3x,k4x,k1y,k2y,k3y,k4y;
      h = tf-ti;
      t = ti;
      k1x = h*d1x(t,xi,yi);
      k1y = h*d1y(t,xi,yi);
      k2x = h*d1x(t+h/2.0,xi+k1x/2.0,yi+k1y/2.0);
      k2y = h*d1y(t+h/2.0,xi+k1x/2.0,yi+k1y/2.0);
      k3x = h*d1x(t+h/2.0,xi+k2x/2.0,yi+k2y/2.0);
      k3y = h*d1y(t+h/2.0,xi+k2x/2.0,yi+k2y/2.0);
      k4x = h*d1x(t+h,xi+k3x,yi+k3y);
      k4y = h*d1y(t+h,xi+k3x,yi+k3y);
      xf = xi + (k1x + 2.0*(k2x+k3x) + k4x)/6.0;
      yf = yi + (k1y + 2.0*(k2y+k3y) + k4y)/6.0;
      return 0.0;
}
```

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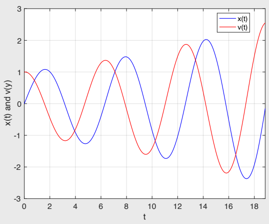
Example: Harmonic oscillator

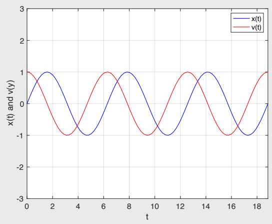
$m\frac{d^2x}{dt^2} = -kx,$ as two first − order ODEs $\frac{dx}{dt} = v,$ $\frac{dv}{dt} = -\frac{k}{m}x$

with initial conditions $x(0) = 0,\ v(0) = 1, k = 1, m = 1$ and step 0.1

Explicit Euler



RKF45



Explicit Euler – no conservation of energy!

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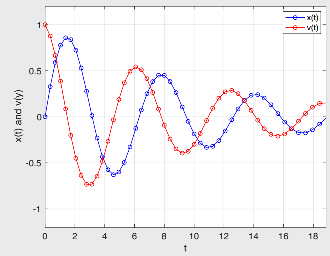
Example: Harmonic oscillator with linear resistance

$m\frac{d^2x}{dt^2} = -kx - a\frac{dx}{dt},$ as $\frac{dx}{dt} = v,$ $\frac{dv}{dt} = -\frac{k}{m}x - \frac{a}{m}v$

with initial conditions $x(0) = 0,\ v(0) = 1, k = 1, m = 1, a = 0.2$

RKF45 with step-size control (tolerance 10^{-5})



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MatLab: RK method for a system of n 1st-order ODEs

```
{%
*==========================================================
 RK4n Solution of a system of n first-order ODE
 method:      Runge-Kutta 4th-order
 Alex G. November, 2020
 ----------------------------------------------------------
 call ... (supplied by a user)
   dx = dnx(n, t, x) functions dx/dt  where dx and x are arrays size n
 input ...
   n     - number of first order equations
   ti    - initial time
   tf    - solution time
   xi    - initial values (array size n)
 output ...
   xf    - solutions (array size n)
==========================================================*/
%}
```

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MatLab: RK method for a system of n 1st-order ODEs

```
function [xf] = RK4n(n,ti, tf, xi)
h = tf-ti;
t = ti;
dx=dnx(n, t, xi);
for j = 1: n
    k1(j) = h*dx(j);
    x(j)  = xi(j) + k1(j)/2.0;
end
dx = dnx(n, t+h/2.0, x);
for j = 1: n
    k2(j) = h*dx(j);
    x(j)  = xi(j) + k2(j)/2.0;
end
dx =  dnx(n, t+h/2.0, x);
for j = 1: n
    k3(j) = h*dx(j);
    x(j)  = xi(j) + k3(j);
end
dx = dnx(n, t+h, x);
for j = 1: n
    k4(j) = h*dx(j);
    xf(j) = xi(j) + k1(j)/6.0+k2(j)/3.0+k3(j)/3.0+k4(j)/6.0;
end
end % end of RK4n
```

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Part 2:

Particle dynamic

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Faster methods for particle dynamics are needed

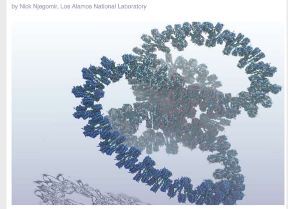
While RK methods very provide excellent accuracy, considerable computation time is needed for systems with MANY particles.

Now molecular dynamics simulations involves up to a billion of particles!

APRIL 23, 2019

Scientists create first billion-atom biomolecular simulation

by Nick Njagomir, Los Alamos National Laboratory



A Los Alamos-led team created the largest simulation to date of an entire gene...

Researchers at Los Alamos National Laboratory have created the largest simulation to date of an entire gene of DNA, a feat that required one billion atoms to model and will help researchers to better understand and develop cures for diseases like cancer.

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Two most popular methods

The leap-frog and Verlet methods are the most popular methods.

Both the leap-frog and Verlet methods use that in Newton's second law

$$m\frac{d^2x}{dt^2} = F(t, x, x')$$

the force depends only on position but not the velocity or time, i.e.

$$F(t, x, x') = F(x)$$

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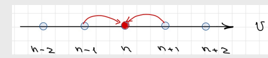
The leapfrog method

Consider Newton's second law

$$\frac{dv}{dt} = v' = \frac{F(x)}{m}, \qquad \frac{dx}{dt} = v.$$

Using n as a base point we can write

$$v_{n-1} = v_n - v_n'\Delta t + \frac{1}{2}v_n''(\Delta t)^2$$



$$v_{n+1} = v_n + v_n'\Delta t + \frac{1}{2}v_n''(\Delta t)^2$$

then taking the difference $v_{n+1} - v_{n-1}$ gives

$$v_{n+1} = v_{n-1} + 2v_n'\Delta t + O(\Delta t)^3$$

$$v_{n+1} = v_{n-1} + \frac{2}{m}F(x_n)\Delta t$$

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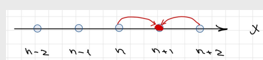
The leapfrog method (cont.)

For x using $n+1$ as a base point

$$x_n = x_{n+1} - x_{n+1}'\Delta t + \frac{1}{2}x_{n+1}''(\Delta t)^2$$

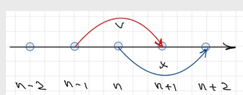


$$x_{n+2} = x_{n+1} + x_{n+1}'\Delta t + \frac{1}{2}x_{n+1}''(\Delta t)^2$$

then

$$x_{n+2} = x_n + 2v_{n+1}\Delta t + O(\Delta t)^3$$

Now both v and x together



$$v_{n+1} = v_{n-1} + \frac{2}{m}F(x_n)\Delta t \ \text{ calculate first}$$

$$x_{n+2} = x_n + 2v_{n+1}\Delta t + O(\Delta t)^3 \ \text{ now update } x$$

We need v_{n-1} to start calculation. We can use backward Euler step

$$v_{n-1} = v_n - \frac{F(x_n)}{m}\Delta t$$

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The leapfrog method - summary

- The leapfrog method is a second-order method
- It is conditionally stable, as long as the time-step Δt is constant
- It conserves (mostly) the energy of dynamical systems in a long run. This is especially useful when computing orbital dynamics, as many other integration schemes, such as 4th order Runge-Kutta method, do not conserve energy and allow the system to drift substantially over time.
- The method is time-reversible, i.e. one can integrate forward n steps, and then reverse the direction of integration and integrate backwards n steps to arrive at the same starting position.
- There are a couple variations of the method.

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The Verlet method (or Störmer–Verlet method)

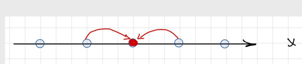
The algorithm was first used by Delambert 1791. It has been rediscovered many times since then.

L. Verlet used it in the 60-s for calculations in molecular dynamics

$$\frac{dv}{dt} = v' = \frac{F(x)}{m}, \qquad \frac{dx}{dt} = v.$$

Using n as a base point we can write

$$x_{n-1} = x_n - x_n'\Delta t + \frac{1}{2}x_n''(\Delta t)^2$$



$$x_{n+1} = x_n + x_n'\Delta t + \frac{1}{2}x_n''(\Delta t)^2$$

$$x_{n+1} - x_{n-1} = 2x_n'\Delta t, \qquad x_n' = (x_{n+1} - x_{n-1})/2\Delta t + O(\Delta t)^3$$

$$x_{n+1} + x_{n-1} = 2x_n + x_n''(\Delta t)^2$$

$$x_n'' = \frac{x_{n+1} - 2x_n + x_{n-1}}{(\Delta t)^2} = \frac{F(x_n)}{m}$$

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The Verlet method (cont.)

$$x_{n+1} = x_n + x_n'\Delta t + \frac{1}{2}x_n''(\Delta t)^2$$

$$x_n' = (x_{n+1} - x_{n-1})/2\Delta t, \qquad x_n'' = \frac{x_{n+1} - 2x_n + x_{n-1}}{(\Delta t)^2} = \frac{F(x_n)}{m}$$

Using the derivatives in the first equation gives

$$x_{n+1} = 2x_n - x_{n-1} + \frac{F(x_n)}{m}(\Delta t)^2 + O(\Delta t)^4$$

and we do not need velocities! But if we need we can use

$$v_{n+1} = \frac{x_{n+2} - x_n}{2\Delta t} \quad \text{one point behind}$$

Attention: we need x_{n-1} to start the run, and we can use

$$x_{n-1} = x_n - v_n\Delta t + \frac{1}{2}\frac{F(x_n)}{m}(\Delta t)^2$$

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The Verlet method- summary

- Global errors: for $x \sim O(\Delta t)^3$, for $v \sim O(\Delta t)^2$
- The method is very popular in computing trajectories in molecular dynamics simulations
- The Verlet method provides good numerical stability
- The method is time-reversible
- There is a velocity Verlet version similar to the leapfrog method.

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Part 3a:

Oscillatory motion and chaos

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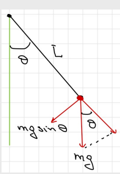
Simple pendulum

Newton's second law for rotational motion of a pendulum

$$I\frac{d^2\vartheta}{dt^2} = \tau_g + \tau_d + \tau_{external}$$

where τ_g is the torque by the gravitational force, τ_d is the torques by the drag force, and $\tau_{external}$ is the external periodic force

For a point-like mass m on a string length L, $I = mL^2$, $\tau_g = -Lmg\sin\vartheta$



$$\frac{d^2\vartheta}{dt^2} = -\frac{g}{L}\sin\vartheta - \frac{\beta}{mL^2}\frac{d\theta}{dt} + \frac{F_{ext}}{mL^2}\cos\omega t$$

or

$$\frac{d^2\vartheta}{dt^2} = -\omega_0^2\sin\vartheta - \alpha\frac{d\theta}{dt} + f_{ext}\cos\omega t$$

$$\omega_0^2 = \frac{g}{L}, \qquad \alpha = \frac{\beta}{mL^2}, \qquad f_{ext} = \frac{F_{ext}}{mL^2}$$

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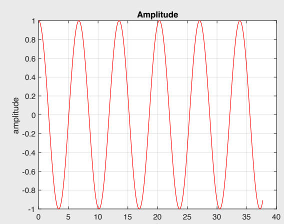
Simple pendulum: simple harmonic motion $\theta \ll 1$

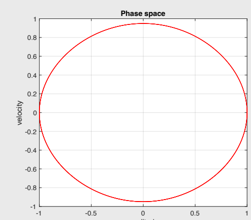
Using the approximation $\sin\vartheta \approx \vartheta$ and setting $\tau_d = 0,\ \tau_{external} = 0$

$$\frac{d^2\vartheta}{dt^2} = -\omega_0^2\sin\vartheta - \alpha\frac{d\theta}{dt} + f_{ext}\cos\omega t \quad \rightarrow \quad \frac{d^2\vartheta}{dt^2} = -\omega_0^2\vartheta$$

Classical harmonic motion

$$\theta(t) = A\cos(\omega_0 t + \varphi), \qquad T = 2\pi/\omega_0$$





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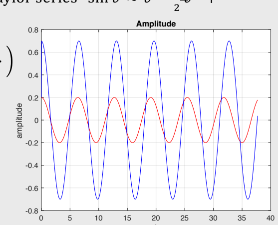
Simple pendulum without small angle approximation

Still disregarding $\tau_d = 0,\ \tau_{external} = 0$

$$\frac{d^2\vartheta}{dt^2} = -\omega_0^2\sin\vartheta - \alpha\frac{d\theta}{dt} + f_{ext}\cos\omega t \quad \rightarrow \quad \frac{d^2\vartheta}{dt^2} = -\omega_0^2\sin\vartheta$$

For $\vartheta_0 = 1.0$ motion is periodic but the period of oscillations is larger than the simple harmonic one since in Taylor series $\sin\vartheta \approx \vartheta - \frac{1}{2}\vartheta^2 + \cdots$

$$T = 2\pi\sqrt{\frac{L}{g}}\left(1 + \frac{1}{16}\vartheta_0^2 + \frac{11}{3072}\vartheta_0^4 + \cdots\right)$$



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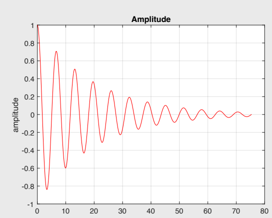
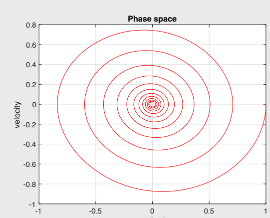
Simple pendulum with dissipation

$$\frac{d^2\vartheta}{dt^2} = -\omega_0^2\sin\vartheta - \alpha\frac{d\theta}{dt}$$

For $\alpha = 0.1$.





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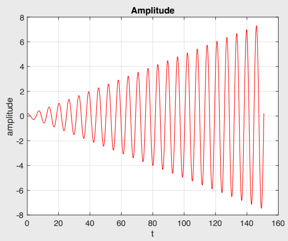
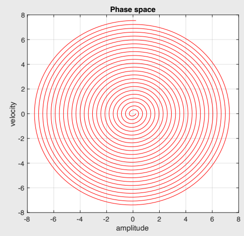
Resonance when $\omega = \omega_0$

$$\frac{d^2\vartheta}{dt^2} = -\omega_0^2\vartheta + f_{ext}\cos\omega t$$

with $\omega = \omega_0$ and $f_{ext} = 0.1$

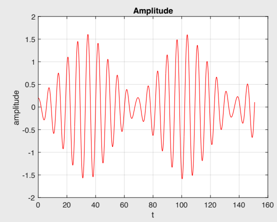


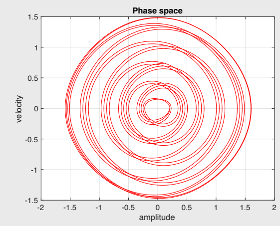


Resonance when $\omega = \omega_0$ but $\sin\vartheta \neq \vartheta$

$$\frac{d^2\vartheta}{dt^2} = -\omega_0^2\sin\vartheta + f_{ext}\cos\omega t$$

with $\omega = \omega_0$ and $f_{ext} = 0.1$



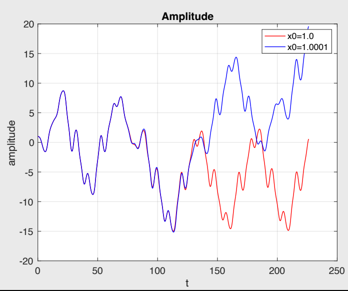


The amplitude does not grow without bound. Do you see why?

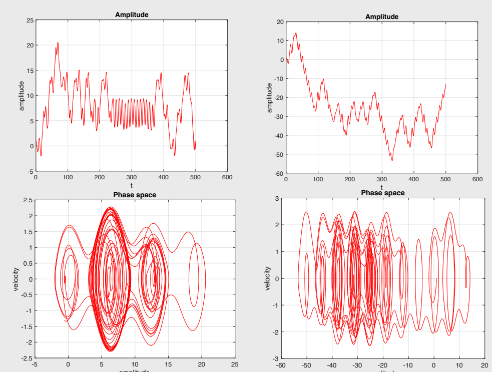
Complex motion in a non-linear system – CHAOS!

$$\frac{d^2\vartheta}{dt^2} = -\omega_0^2\sin\vartheta - \alpha\frac{d\theta}{dt} + f_{ext}\cos\omega t$$

High sensitivity to the initial condition when all there forces are in play.

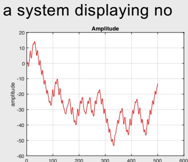


Complex motion in a non-linear system – CHAOS!



Complex motion in a non-linear system – CHAOS!

- A chaotic system is one with an extremely high sensitivity to parameters or initial conditions
- The sensitivity to even miniscule changes is so high that, in practice, it is impossible to predict the long range behavior unless the parameters are known to infinite precision (which they never are in practice)
- Chaotic motion is not random
- Chaos is the deterministic behavior of a system displaying no discernable regularity
- Note: a double pendulum is a good system to study chaos



Chaotic structure in phase space

Limit cycles

When a chaotic pendulum is driven by a not-too-large driving torque, it is possible to pick the magnitude for this torque such that after the initial transients die off, the average energy put into the system during one period exactly balances the average energy dissipated by friction during that period.

This leads to *limit cycles* that appear as closed ellipse-like figures.

(Yet unstable solutions may make sporadic jumps between limit cycles.)

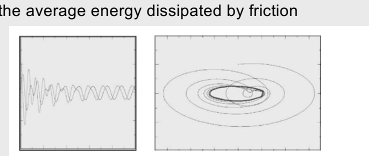
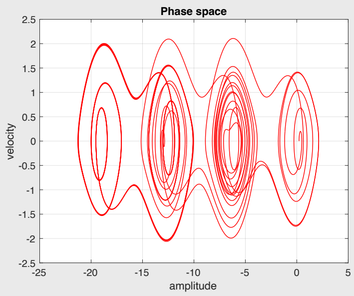


Figure 15.5 (a) Position vs. time for two initial conditions of a chaotic pendulum that end up with the same limit cycle. (b) A phase space plot of position versus velocity for the limit cycle shown in (a) (courtesy of W. Hager).

* from Landau et al. Computational Physics (2015)

Chaotic structure in phase space

Limit cycles: Four dominant cycles



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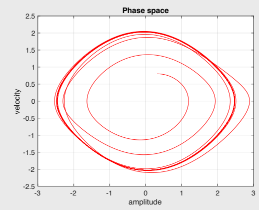
Chaotic structure in phase space

Predictable attractors

There are orbits, such as fixed points and limit cycles, into which the system settles or returns to often, and that are not particularly sensitive to initial conditions. If your location in phase space is near a predictable attractor, ensuing times will bring you to it.



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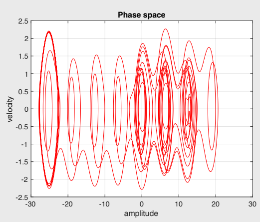
Chaotic structure in phase space

Strange attractors

Well-defined, yet complicated, semi-periodic behaviors that appear to be uncorrelated with the motion at an earlier time. They are distinguished from predictable attractors by being fractal chaotic, and highly sensitive to the initial conditions. Even after millions of oscillations, the motion remains attracted to them.



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Chaotic structure in phase space

Chaotic paths

Regions of phase space that appear as filled-in bands rather than lines. Continuity within the bands implies complicated behaviors, yet still with simple underlying structure.

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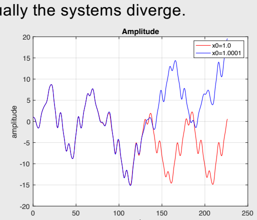
Butterfly effect

One of the classic remarks about the hypersensitivity of chaotic systems to the initial conditions is that the weather pattern in North America is hard to predict well because it is sensitive to the flapping of butterfly wings in South America.

Although this appears to be counterintuitive because we know that systems with essentially identical initial conditions should behave the same, eventually the systems diverge.



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The Lorenz model

In 1962 Lorenz was looking for a simple model for weather predictions and simplified the heat-transport equations to the three equations.

$$\frac{dx}{dt} = 10(y - x)$$

$$\frac{dy}{dt} = -xz + 28x - y$$

$$\frac{dz}{dt} = xy - \frac{8}{3}z$$

The solution of these simple nonlinear equations gave the complicated behavior that has led to the modern interest in chaos!

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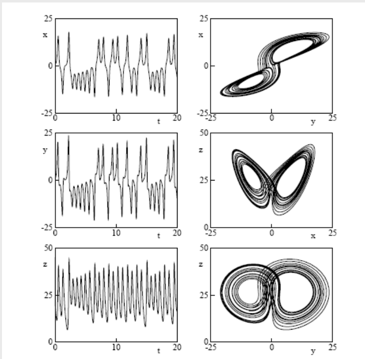
Examples: solutions to the Lorenz model



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Hamiltonian chaos

When a number of degrees of freedom becomes large, the possibility of chaotic behavior becomes more likely.

Examples:
The solar system, particles in EM fields, the rings of Saturn, …

Attention: no dissipation!

Constants of motion: Energy, Momentum (linear, angular)

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Lyapunov exponent

How can we quantify this lack of predictably?

This divergence of the trajectories can be described by the Lyapunov exponent λ, which is defined by the relation

$$|\Delta x_n| = |\Delta x_0| e^{\lambda n}$$

where Δx_n is the difference between the trajectories at time n. If the Lyapunov exponent λ is positive, then nearby trajectories diverge exponentially.

Chaotic behavior is characterized by the exponential divergence of nearby trajectories.

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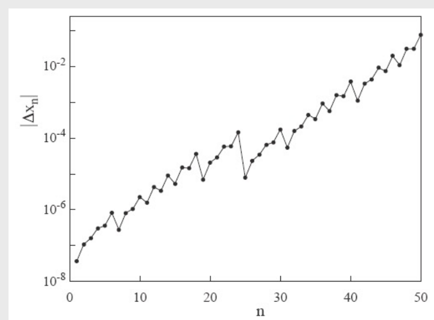
Lyapunov exponent. $|\Delta x_n| = |\Delta x_0| e^{\lambda n}$

Example



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"Control your chaos" from the movie - The Witcher

- The dream of classical physics was that if the initial conditions and all the forces acting on a system were known, then we could predict the future with as much precision as we desire.
- The existence of chaos has shattered that dream.
- However, if a system is chaotic, we can still control its behavior with small, but carefully chosen, perturbations of the system.
- Good illustration can be found in Gould et all, Computer simulation methods. Application to physical systems (2007)

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Part 3a:

Projectile motion

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2D (2-dimensional) projectile motion

Forces: gravity, drag force and potentially magnus force (spin related)

$m\frac{d^2x}{dt^2} = F_{Dx}$ or $\frac{dx}{dt} = v_x$, $m\frac{dv_x}{dt} = F_{Dx}(v_x, v_y)$

$m\frac{d^2y}{dt^2} = -mg + F_{Dy}$ or $\frac{dy}{dt} = v_y$, $m\frac{dv_y}{dt} = -mg + F_{Dy}(v_x, v_y)$

Initial value problem:

$$x(0) = x_0, \quad x'(0) = v_x(0) = v_{x0}$$
$$y(0) = y_0, \quad y'(0) = v_y(0) = v_{y0}$$

The system of two second-order ODEs can be rewritten as a system of four first-order ODEs.

All the methods studied before for solving first-order ODEs can be used here.

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The drag force

The drag force $\vec{F}_D$ and the velocity $\vec{v}$ point in opposite directions

$$\vec{F}_D = -f(v)\hat{v},$$

where $\hat{v}$ is the unit vector in the direction of velocity, and $f(v)$ is the magnitude of the drug force.

The function $f(v)$ that give the magnitude of the air resistance varies with v in a complicated way, however often it can be well approximated as*

$$f(v) = bc + cv^2 = f_{lin} + f_{quad}$$

In many practical cases we will work with the quadratic drag component

The physical origin of the terms: The linear term corresponds to the viscosity drag of the medium. The quadratic term describes the acceleration of the mass of air pushed by the projectile.

*for more details see Classical mechanics by J.R Taylor (Chapter 2)

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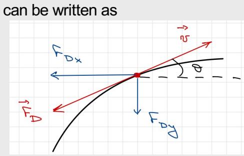
The drag force (cont.)

In the (x, y) plane the quadratic drag force can be written as



$F_{Dx} = -cv^2\cos\theta$, $F_{Dy} = -cv^2\sin\theta$

where $v^2 = v_x^2 + v_y^2$

Since $\cos\theta = v_x/v$, $\sin\theta = v_y/v$

$F_{Dx} = -cv_x v$

$F_{Dy} = -cv_y v$

and c is a coefficient that is often approximated as

$$c = \frac{1}{2}C\rho A$$

where C is the drag coefficient (dimensionless) depending on an object shape and can be determined by wind tunnel measurements. For many objects it can be approximated by a value within 0.05 – 1.0. A is is the cross sectional area.

ρ is the density of the air. Since air density varies with altitude, one may approximate it as $\rho = \rho_0\exp(-y/y_0)$ where ρ_0 is the density at sea level $(y = 0)$ and $y_0 \approx 10{,}000$m.

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Terminal speed

from

$$mg = \frac{1}{2}C\rho A v_t^2$$

object	speed (m/s)	speed (mph)	distance (m) 95%
shot	145	316	2500
sky diver	60	130	430
baseball	42	92	210
basketball	20	44	47
raindrop	7	15	6
parachutist	5	11	3

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The drag force (cont.)

$c = \frac{1}{2}C\rho A$

Generally the coefficient C depends depends on speed v (aerodynamic drag crisis).

Example for baseball: Frohlich Am J. Phys. 52, 325 (1984).

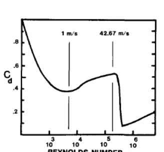

Fig. 1. Drag coefficient C_d of a smooth sphere versus Reynolds number R (Refs. 16, 10). Note especially the "drag crisis," the sharp decrease in drag that occurs at about $R = 4\times10^5$. The vertical lines correspond to velocities of 1 m/s and 42.67 m/s for a smooth sphere with the diameter of a baseball. 42.67 m/s is the terminal velocity of a baseball as measured by Briggs,[7] and is approximately the velocity of the fastest major league pitchers. Most previous investigators have assumed that a baseball is aerodynamically smooth, and that $C_d = 0.5$ for all Reynolds numbers.

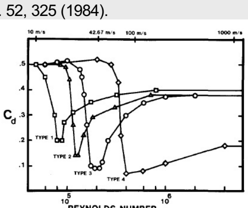

Fig. 2. Effect of the surface roughness of a sphere on drag coefficient C_d at Reynolds numbers near the drag crisis. Surface roughness is parameterized by k/d where k is the height of the roughness elements, and d is the sphere diameter. The roughness for the various spheres shown are type 1 ball—$k/d = 1250\times10^{-5}$; type 2 ball—$k/d = 500\times10^{-5}$; type 3 ball—$k/d = 150\times10^{-5}$; type 4 ball—smooth sphere; and type 5 ball—$C_d = 0.5$ for all R. The top horizontal scale shows the equivalent air speed for a sphere with the diameter of a baseball. The plotted symbols are from Achenbach.[16] The lines connecting measured values show values used for C_d for the calculations reported in this paper.

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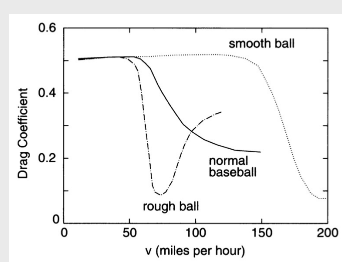
Physics of baseball

*Physics of baseball – from R.K. Adair, The physics of baseball



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The equations of motion

For motion in (x, y) plane

$$m\frac{d^2x}{dt^2} = -cv_x v$$

$$m\frac{d^2y}{dt^2} = -mg - cv_y v$$

where m is the mass of the object, g is the free-fall acceleration, and c is the drag coefficient.

Note: a good test for numerical solutions is to compare with analytic solutions for $c = 0$.

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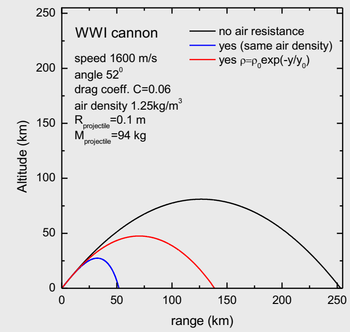
Example: the effect of air resistance



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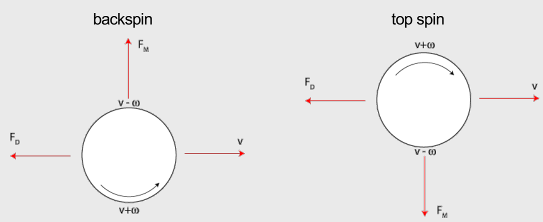
The effect of spin (Magnus force)

Force on a spinning object moving through air can be approximated as (Magnus force)

$$\vec{F}_M = S(v)\,\vec{\omega}\times\vec{v}$$



A common approximation: $F_M = S_0\omega v$, where the coefficient S_0 can be found elsewhere.

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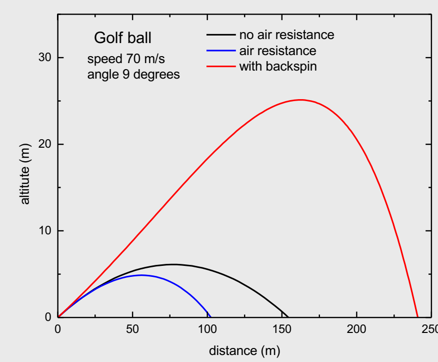
Example



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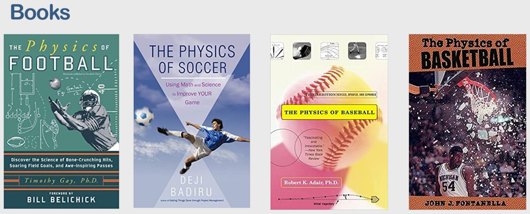
Books



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Part 3b:

Classical scattering

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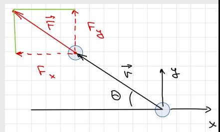
Classical scattering on one center

Consider scattering a projectile on a potential center. Let the force on the projectile from the target to be Coulomb force



$$\vec{F} = k\,\frac{Z_p Z_t}{r^2}\hat{r}$$

where the notations are obvious. Since $F_x = F\cos\vartheta$, $F_y = F\sin\vartheta$ and $\cos\vartheta = x/r$, $\sin\vartheta = y/r$, with $r^2 = x^2 + y^2$. Then

$$m\frac{d^2x}{dt^2} = k\frac{Z_p Z_t}{r^3}x$$

$$m\frac{d^2y}{dt^2} = k\frac{Z_p Z_t}{r^3}y$$

A trajectory can be evaluated for given initial conditions: x_0, v_{x0}, y_0, v_{y0}. Use conservation of energy and angular momentum as a test.

$$E = mv^2/2 + kZ_pZ_T/r, \qquad L_z = m(xv_y - yv_x).$$

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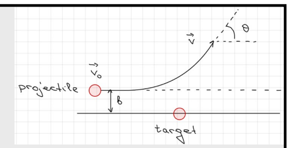
Differential cross section



There are two key parameters of the collisional theory:

The impact parameter b is defined as their perpendicular distance from the projectiles incoming straight line path to parallel axis through the target's center.

The scattering angle θ is defined as the angle between the incoming and outgoing velocities of the projectile.

The differential cross section can be calculated from

$$\frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$$

Calculations can be tested by using the analytic solution (Rutherford scattering)

$$\theta = 2\,\mathrm{atan}\left(\frac{kZ_pZ_t}{bmv_0^2}\right) \qquad \frac{d\sigma}{d\Omega} = \left(\frac{kZ_pZ_t}{4K\sin^2(\theta/2)}\right)^2 \qquad K = \frac{mv_0^2}{2}$$

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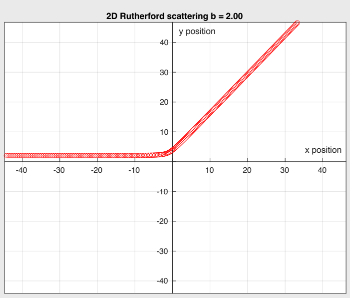
Rutherford scattering on heavy target

```
Rutherford angle (analytical)   =   52.89 deg
Angle projectile (numerical)    =   52.23 deg
```



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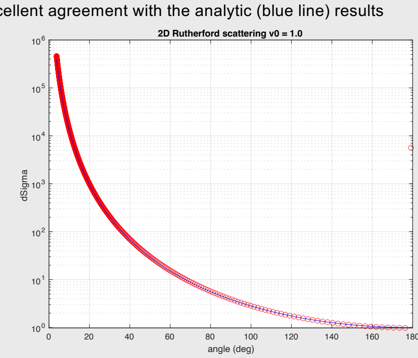
Differential cross section

Excellent agreement with the analytic (blue line) results



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Classical scattering on a light target

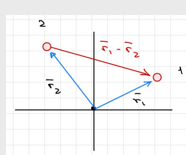
In case of scattering on a light target (when the target can move) equations of motion are

$$m_1\frac{d^2x_1}{dt^2} = k\frac{Z_1Z_2}{r^3}(x_1 - x_2)$$

$$m_1\frac{d^2y_1}{dt^2} = k\frac{Z_1Z_2}{r^3}(y_1 - y_2)$$

$$m_2\frac{d^2x_2}{dt^2} = k\frac{Z_1Z_2}{r^3}(x_2 - x_1)$$

$$m_2\frac{d^2y_2}{dt^2} = k\frac{Z_1Z_2}{r^3}(y_1 - y_2)$$



Where m_1 and m_2 are masses of the projectile and target, etc.

$$r = |r_{1,2}| = [(x_1 - x_2)^2 + (y_1 + y_2)^2]^{1/2}$$

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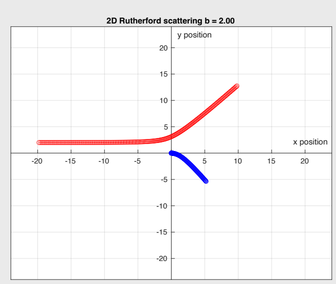
Scattering on light target (recoil)



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Scattering on two centers (or more centers)

Let a projectile (particle 1) scatters on two centers (2 and 3)

$$m_1 \frac{d^2 x_1}{dt^2} = k\,\frac{Z_1 Z_2}{r_{1,2}^3}(x_1 - x_2) + k\,\frac{Z_1 Z_3}{r_{1,3}^3}(x_1 - x_3)$$

$$m_1 \frac{d^2 y_1}{dt^2} = k\,\frac{Z_1 Z_2}{r_{1,2}^3}(y_1 - y_2) + k\,\frac{Z_1 Z_3}{r_{1,3}^3}(y_1 - y_3)$$

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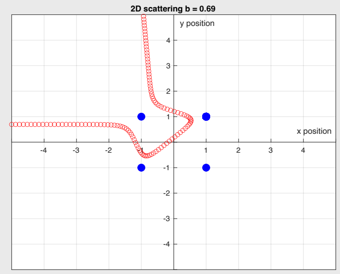
Scattering on four centers



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Part 3c:

Planetary and satellite motion

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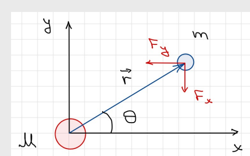
Gravitational force

Newton's universal law of gravitation states that a particle of mass M attracts another particle of mass m with a force given by

$$\vec{F} = -G\,\frac{mM}{r^3}\hat{r}$$

where the vector $\hat{r}$ is directed from M to m. The negative sign in implies that the gravitational force is attractive. And G is the gravitational constant.



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Properties of the gravitational force

1. Central force

The gravitational force has two general properties: its magnitude depends only on the separation of the particles, and its direction is along the line joining the particles. Such a force is called a central force. The assumption of a central force implies that the orbit of the Earth (if m is Earth and M is the Sun) is restricted to a plane (x, y), and the angular momentum is conserved

$$L_z = m(xy_y - yv_x)$$

2. Total energy is conserved

$$E = \frac{1}{2}mv^2 - G\,\frac{mM}{r}$$

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Equations of motion

If we fix the coordinate system at the mass M, the equation of motion of of mass m is

$$m\,\frac{d^2\vec{r}}{dt^2} = -G\,\frac{mM}{r^3}\hat{r}$$

It is convenient to write the forces and equations of motion in Cartesian coordinates with $r^2 = x^2 + y^2$

$$F_x = -G\,\frac{mM}{r^2}\cos\vartheta = -G\,\frac{mM}{r^3}x$$

$$F_y = -G\,\frac{mM}{r^2}\sin\vartheta = -G\,\frac{mM}{r^3}y$$

$$\frac{d^2x}{dt^2} = -G\,\frac{M}{r^3}x$$

$$\frac{d^2y}{dt^2} = -G\,\frac{M}{r^3}y$$

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Circular orbits

For circular orbits

$$\frac{mv^2}{r} = -G\frac{mM}{r^2}$$

Then for the speed and period

$$v = \left(\frac{GM}{r}\right)^{1/2}, \qquad T = \frac{2\pi r}{v}$$

It is much more convenient to work with astronomical units instead of the SI units. Thus, for the solar system we can introduce the unit of distance as 1 AU = distance to the Sun, and the unit of time as 1 year. Then with $r = 1$ and $T = 1$ we have $v = 2\pi$ and $GM = 4\pi^2$.

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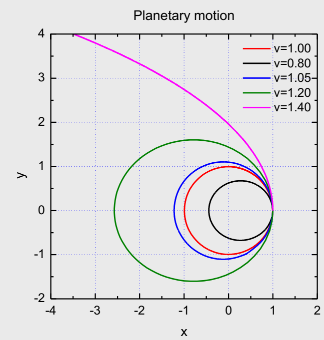
Examples

$v = 1$ on the figure corresponds to $v = 2\pi$



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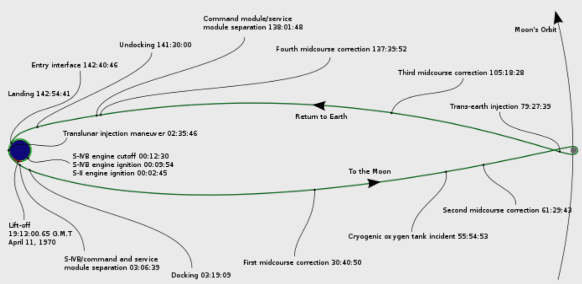
Examples: Apollo 13 (the mission to the Moon)

Flightpath



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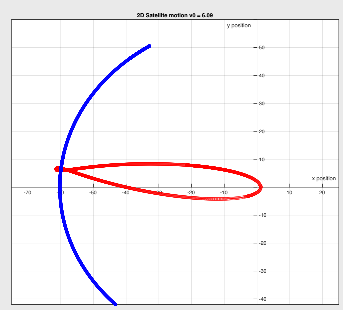
Examples: Apollo 13 (the mission to the Moon)

simulation



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