Operators

Operator can be represented by its matrix elements.
$\Omega_{ij} =< i|\Omega|j >$
$\Omega\colon \mathbf{V} \rightarrow \mathbf{V}$

- Hermitian
  $\Omega = \Omega^\dagger$; $\Omega^\dagger_{ij} = \Omega^*_{ji}$
- Unitary
  $U\colon \mathbf{V} \rightarrow \mathbf{V}$
  $U^\dagger U = UU^\dagger = \mathbf{1}$
  $\langle Uw|Uv\rangle = \langle w|U^\dagger U|v\rangle = \langle w|\mathbf{1}|v\rangle = \langle w|v\rangle$ operator acting on vectors w and v
  $\delta_{jk} = \sum_i U^*_{ji} U_{ik}$
- Unit Matrix
  Hermitian and Unitary

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1}$$

- Projection Operator $\mathbf{P}_{\mathbf{V}'}$
  Generally takes a vector and removes some but not all components. Projects component of vector on subspace, $\mathbf{V'}$ along that direction
  $\mathbf{V'} \subset \mathbf{V}$
  If subspace has just 1 dimension (multiples of some vector $|v'\rangle \in \mathbf{V'}$) $\Longrightarrow \mathbf{P}_{\mathbf{V}'} = |v'\rangle\langle v'|$
  $\mathbf{P}_{\mathbf{V}'}|v\rangle = |v'\rangle\langle v'|v\rangle$
  Special case of projection on basis vectors $\mathbf{P}_j = |j\rangle\langle j|$
  Any vector can be written as $|v\rangle = \sum_j P_j|v\rangle \therefore \sum_j \mathbf{P}_j = \mathbf{1}$
  if dimension $\mathbf{V'} > 1$
  1.find orthonormal base of $\mathbf{V'}$: $|j'\rangle, j' = 1..m < n$
  2.find $\mathbf{P}_{\mathbf{V}'} = \sum_{j'} |j'\rangle\langle j'|$ vector projects onto $\mathbf{V'}$
  if we take $1 - \mathbf{P}_{\mathbf{V}'}$ we project onto orthogonal subspace $V \rightarrow \mathbf{V}'_\perp$
  Projection operators are hermitian but NOT unitary
  Projection operator acting on itself gives same back: $\mathbf{P}_{\mathbf{V}'}\mathbf{P}_{\mathbf{V}'} = \mathbf{P}_{\mathbf{V}'}$

Eigenvalues and Eigenvectors

-examples of projection operator
$\mathbf{P}_j$ has eigenvalues of 1 and 0, with eigenfunctions of $|j\rangle$ and $|i\rangle$ for $i \neq j$
For an arbitrary projection operator $\mathbf{P}_{\mathbf{V}'}$ any vector in subspace that this projects onto has eigenvalue of 1; and any vector in the orthogonal subspace $\mathbf{V}'_\perp$ is eigenvector with eigenvalue 0
-unit matrix: eigenvalue of 1; degenerate, n linearly independent eigenvectors with the same eigenvalue
any operator that is unitary or hermitian (or. in general, commutes with its adjoint) has exactly n linearly independent EF (eigenvectors, also called "eigenfunctions")

-counterintuitive example in 2D complex vector space:

$$\Omega = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \tag{2}$$

unitary but not Hermitian

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \omega \begin{bmatrix} a \\ b \end{bmatrix} \tag{3}$$

where ω is a constant
this example has no degenerate eigenvalues
$(\Omega - \omega\mathbf{I})|\omega\rangle = 0$

singular operator/matix det must be 0

$$\det \begin{bmatrix} \cos\theta - \omega & \sin\theta \\ -\sin\theta & \cos\theta - \omega \end{bmatrix} = (\cos\theta - \omega)^2 + \sin^2\theta = 0 \tag{4}$$

we solve this equation for ω

$$\cos\theta - \omega = \pm i \sin\theta \Rightarrow \omega_{1,2} = e^{\pm i\theta}$$

$\therefore$ this does have eigenvalues, but they were hard to see quickly as they are complex

Finding eigenvectors

have "n" zeros, but many can be repeats

$$(\Omega - \omega \mathbf{I}) = Pol^n(\omega)$$

How to find eigenvectors? (back to example...)

$$\omega_1 : (\cos\theta - e^{i\theta})a + \sin\theta b = 0$$
$$\omega_2 : -\sin\theta a + (\cos\theta - e^{i\theta})b = 0$$

after solving, we find: $b = \pm ia$

$\therefore$ solution is

$$\begin{bmatrix} a \\ ia \end{bmatrix} \tag{5}$$

now we need to normalize our results

let $a = \frac{1}{\sqrt{2}}$

$$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ i \end{bmatrix} \tag{6}$$

and our other eigenvector, $e^{-i\theta}$

$$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -i \end{bmatrix} \tag{7}$$

In general, we can always find solutions to the characteristic polynomial $det(\Omega - \omega\mathbf{I}) = Pol^n(\omega)$ (in the complex numbers). If we have 1 solution, ω_1, we must be able to find EF $|\omega_1, 1\rangle$ by solving set of $(n-1)$ linear equations. (The second index "1" is only needed in case the eigenvalue is degenerate). We assume $|\omega_1, 1\rangle$ is normalized. Now we must find all vectors perpendicular to this vector, which form a new subspace. $\mathbf{V} = \mathbf{V}_{|\omega_1\rangle} + \mathbf{V}_{|\perp\omega_1\rangle}$ Next we find orthonormal basis of $\mathbf{V}_{|\perp\omega_1\rangle}$ which, together with $|\omega_1, 1\rangle$ forms a new basis. In this basis, Ω has a "block diagonal form" which acts on $\mathbf{V}_{|\omega_1\rangle}$ as a simple multiplication with ω_1 and on $\mathbf{V}_{|\perp\omega_1\rangle}$ as a regular $(n-1) \times (n-1)$ matrix. We can now repeat this procedure to reduce the size of *that* matrix, until we are left with a complete, orthonormal new basis $|\omega_i, \alpha\rangle$ in which the operator Ω becomes purely diagonal, with its eigenvalues on the diagonal.
Returning to our prior example:

$$\Omega = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \tag{8}$$

$$\begin{bmatrix} a \\ b \end{bmatrix} = \frac{a - ib}{\sqrt{2}} |\omega_1\rangle, \frac{a + ib}{\sqrt{2}} |\omega_2\rangle \tag{9}$$

we can write any vector as a linear combination of these

check:

$$\langle\omega_2|\omega_1\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \end{bmatrix} \begin{bmatrix} 1 \\ i \end{bmatrix} = 0 \tag{10}$$

Note: Eigenvalues of hermitian operators are all real, while eigenvalues of unitary operators are all of form $e^{i\phi}$.