Tuesday Lecture (11/29/2011)

Reminder

Consider two vector spaces $\mathbb{V}$ and $\mathbb{U}$. We want to define the "direct product space" $\mathbb{V} \otimes \mathbb{U}$.

We take any two basis elements from $\mathbb{V}$ and $\mathbb{U}$, $|V_i, U_j > = |V_i > \otimes |U_j >$, as a basis state of $\mathbb{V} \otimes \mathbb{U}$.

Assume $\mathbb{V}$ has 2 dimensions and $\mathbb{U}$ has 3 dimensions.

$$|V_i > = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$|U_i > = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Then the basis elements will be,

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \text{ etc. (6 in total)}$$

General state,

$$|\psi_{\mathbb{V}\otimes\mathbb{U}} > = \sum_{ij} \alpha_{ij} |V_i > \otimes |U_j >$$

in general, can **not** be written as product of just two states from $\mathbb{V}$ and $\mathbb{U}$:

$$|\psi_{\mathbb{V}\otimes\mathbb{U}} > \neq |\psi_{\mathbb{V}} > \otimes |\psi_{\mathbb{U}} >$$

Example: Considering the combination

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right]$$

there is no way to write it as just a single product $\begin{pmatrix} a \\ b \end{pmatrix} \otimes \begin{pmatrix} c \\ d \\ e \end{pmatrix} = \begin{pmatrix} ac \\ ad \\ ae \\ bc \\ bd \\ be \end{pmatrix}$ of states from $\mathbb{V}$ and $\mathbb{U}$.

Another Example:

Let $\mathbb{V} = \mathbb{V}_l$ be the space of solutions with fixed angular momentum l to the radial part of the Schrödinger equation. Let $\mathbb{U}$ be the space of solutions to the angular Schrödinger equation $|l, m>$ for the same angular momentum l, with magnetic quantum number $-l \leq m \leq l$.

For free states, Dim $\mathbb{V} = \mathbb{R}$; for bound states Dim $\mathbb{V} = \mathbb{Z}$. Dim $\mathbb{U} = 2l + 1$. Basis vectors of $\mathbb{V} \otimes \mathbb{U}$ are

$$|\psi_{Elm}> = |R_{El}> \otimes |l, m>$$

$$<\vec{r}|\psi_{Elm}> = R_{El}(r) Y_l^m(\theta, \varphi)$$

Assume a Hamiltonian H that commutes with L^2

$$[H, L^2] = 0$$

Here $l \rightarrow l'$ transition is not possible (if $l \neq l'$). Therefore, all solutions of the Schrödinger equation can be chosen to be Eigenvectors to L^2 with fixed l (or linear combinations thereof).

Any wave function fulfilling the (time-dependent) Schrödinger equation in this subspace with fixed l can be written as

$$|\psi_l>(t) = \int \sum_m a(E, m) \left|R_{El}> \otimes |l, m> e^{-iEt/\hbar} dE\right.$$

Since $\mathbb{U}$ can be thought of consisting of column vectors of $2l + 1$ complex numbers, the most general vector in $\mathbb{V} \otimes \mathbb{U}$ is of the form

$$|\psi> = \begin{pmatrix} R_l(r) \\ R_{l-1}(r) \\ \vdots \\ . \\ R_{-l}(r) \\ \end{pmatrix}$$

For the previous wave function $|\psi_l>(t)$, $R_l(r) = \int a(E, m = l) R_{El}(r) e^{-iEt/\hbar} dE$ etc.

In this interpretation, the basis elements look as follows: $\begin{pmatrix} 0 \\ 0 \\ R_{E,l}(r) \\ \vdots \\ 0 \end{pmatrix}$ etc.

Previously, we have shown that the angular momentum operators $\mathbf{J}^2$, $\mathbf{J}_z$ allow not only integer, but also half-integer values for the quantum numbers j, m where the eigenvalues of $\mathbf{J}^2$ are $j(j+1)\hbar$ and the eigenvalues for $\mathbf{J}_z$ are $m\hbar, -j \leq m \leq j$ (in integer increments).

For each value of j we define a (2j+1) – dimensional subspace with basis $|j, m>$. How do we interpret the half-integer values of j? It turns out that in addition to orbital angular momentum $\vec{\mathbf{L}}$ (which can only have integer values for l), there is also an intrinsic property of each (elementary) particle called **spin** s (somewhat akin to rotation of a body around its own axis). In fact, any Hamiltonian that is consistent with special relativity must commute with this quantity for elementary particles: $[H, S^2] = 0$. Other than charge, the only absolute invariants for elementary particles are their mass m and spin s, which therefore serve to define them.

Particle (elementary ones are **bold**)	Spin s
Higgs , π, K, ^{4}He	0
ν, μ, e, quarks, p, ^{3}He	½
γ, W, **Z**, ρ	1

Each such particle therefore must “live in” a sub space of defined spin

s = 0, +1/2, +1, +3/2, …

with possible basis states $|m_s>, -s \leq m_s \leq s.$

Similar to orbital angular momentum operators, here we have spin operators $S_x, S_y, S_z, S^2, S_+, S_-$ which represent (infinitesimal) rotations in this new space and fulfill all the usual commutator rules as well as relationships like

$$S_z\, S_+ |m_s> = \hbar(m_s + 1)S_+|m_s>$$

In general, a particle with spin s must then be represented in the product space of its spatial coordinates, $\mathbb{V}$ with Dim $\mathbb{V} = \mathbb{R}^3$, and its “spin coordinates”, $\mathbb{U}$, with Dim $\mathbb{U} = 2s + 1$.

The basis states in $\mathbb{V} \otimes \mathbb{U}$ are given by $|\alpha, m> = |\alpha> \otimes |m_s>$; m_s = +s,….,-s (α represents any quantum numbers describing the basis states in spatial coordinates):

$$<\vec{r}|\alpha, m_s> = R_\alpha(\vec{r}) \otimes |m_s>$$

$$\text{Most general state} = \begin{pmatrix} R_s(\vec{r}) \\ R_{s-1}(\vec{r}) \\ \vdots \\ R_{-s}(\vec{r}) \end{pmatrix}$$

If $\mathbf{H} = \mathbf{H}_{\text{spatial}} + \mathbf{H}_{\text{spin}}$, $[\mathbf{H}_{\text{spatial}}, \vec{S}] = 0$ and $\mathbf{H}_{\text{spin}}$ acts only on spin degrees of freedom, then all eigenstates of **H** can be chosen in the form $|\alpha > \otimes |m_s >$.

<u>The simplest non-trivial case is spin-1/2:</u>

S= ½ which yields Dim $\mathbb{U} = 2$, so $\mathbb{U} = \mathbb{C}^2$ with basis states $\binom{1}{0}$, $\binom{0}{1}$

which are eigenfunctions to S_z with magnetic quantum numbers m_s = + ½ and - ½:

$S_z \binom{1}{0} = \frac{\hbar}{2} \binom{1}{0}, S_z \binom{0}{1} = -\frac{\hbar}{2} \binom{0}{1}$ (*)

General $|\psi > \in \mathbb{U}$ => $\binom{\alpha}{\beta}$; $\alpha, \beta \in \mathbb{C}$

If we normalize the vector, then $|\alpha|^2 + |\beta|^2 = 1 \Longrightarrow$ we can write $|\alpha| = \cos\gamma$, $|\beta| = \sin\gamma$

$$\Longrightarrow |\psi \geq \begin{pmatrix} \cos\gamma & e^{i\delta_\alpha} \\ \sin\gamma & e^{i\delta_\beta} \end{pmatrix}$$

$$= e^{i\frac{\delta_\alpha+\delta_\beta}{2}} \begin{pmatrix} \cos\gamma & e^{\frac{i}{2}(\delta_\alpha-\delta_\beta)} \\ \sin\gamma & e^{\frac{-i}{2}(\delta_\alpha-\delta_\beta)} \end{pmatrix}$$

$$= e^{i\frac{\delta_\alpha+\delta_\beta}{2}} \begin{pmatrix} \cos\gamma & e^{\frac{i}{2}\Delta\delta} \\ \sin\gamma & e^{\frac{-i}{2}\Delta\delta} \end{pmatrix}$$

Any operator in this vector space must be represented by a 2x2 matrix:

$\hat{O} = \begin{pmatrix} O_{\frac{1}{2}\frac{1}{2}} & O_{\frac{1}{2}\frac{-1}{2}} \\ O_{\frac{-1}{2}\frac{1}{2}} & O_{\frac{-1}{2}\frac{-1}{2}} \end{pmatrix}$ which can be expressed as a linear combination of 4 "basis" matrices:

$$\hat{O} = \sum_{i=0,x,y,z} \theta_i \sigma_i$$

$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

In particular, the three components of the spin vector operator are

$S_i = \frac{\hbar}{2}\sigma_i$ (i = x,y,z). This can be proven as follows:

$S_z = \frac{\hbar}{2}\sigma_z$ follows simply from the definition of the basis, Eq. (*)

Similarly, it must be true that $S_+ = \hbar\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}$ since $S_+\binom{0}{1} = \hbar\binom{1}{0}$ according to our results for arbitrary j, and $S_- = \hbar\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}$. Then all we need is that $S_x = \frac{1}{2}(S_+ + S_-)$ and $S_y = \frac{1}{2i}(S_+ - S_-)$

Finally, $S^2 = \frac{3}{4}\hbar^2\,\mathbb{1} = s(s+1)\hbar^2\sigma_0$. This does not give anything new. It commutes with all possible operators as it must.

Some properties of the Pauli matrices:

$$\sigma_i\sigma_j = i\sum_k \varepsilon_{ijk}\sigma_k + \delta_{ij}\sigma_0$$

$\sigma_i\sigma_j = -\sigma_j\sigma_i$ => The Pauli matrices anti-commute.

$[\sigma_i, \sigma_j] = 2i\sum_k \varepsilon_{ijk}\sigma_k$

Eigen functions of S_z; $\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}$

S_x => $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$

S_y => $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix}$

As shown above, up to a constant phase (irrelevant), any properly normalized state can be written like

$$\begin{pmatrix}\cos\gamma & e^{-\frac{i}{2}\Delta\delta}\\ \sin\gamma & e^{\frac{i}{2}\Delta\delta}\end{pmatrix}$$

A rotation around the $\hat{n}$ is given by $e^{-\frac{i\theta\hat{n}.\vec{S}}{\hbar}} = \mathbb{1} - i\theta\frac{\hat{n}.\vec{S}}{\hbar} + (-i\theta)^2\frac{\hat{n}.\vec{S}}{\hbar}\frac{\hat{n}.\vec{S}}{\hbar} + \cdots$

We know that $\frac{\hat{n}.\vec{S}}{\hbar} = \frac{\hat{n}\cdot\vec{\sigma}}{2}$

$$e^{-\frac{i\theta\hat{n}.\vec{S}}{\hbar}} = \mathbb{1} - i\theta\frac{\hat{n}\cdot\vec{\sigma}}{2} + \frac{1}{2}\frac{(-i\theta)^2}{2}(\hat{n}\cdot\vec{\sigma})^2 + \cdots$$

$$(\hat{n}\cdot\vec{\sigma})^2 = (\hat{n}_x\,\sigma_x + \hat{n}_y\,\sigma_y + \hat{n}_z\,\sigma_z)(\hat{n}_x\,\sigma_x + \hat{n}_y\,\sigma_y + \hat{n}_z\,\sigma_z)$$

$$= (\hat{n}_x{}^2\sigma_x{}^2 + \hat{n}_y{}^2\sigma_y{}^2 + \hat{n}_z{}^2\sigma_z{}^2) = \mathbb{1}$$

$$= \sum_{odd\ n} \frac{1}{n!}\left(\frac{-i\theta}{2}\right)^n \hat{n}\cdot\vec{\sigma} + \sum_{even\ n} \frac{1}{n!}\left(\frac{-i\theta}{2}\right)^n \mathbb{1}$$

$$= -i \sin{}^{\theta}/_{2}\, \hat{n}\cdot\vec{\sigma} + \cos{}^{\theta}/_{2}\, \mathbb{1}$$

$$= \begin{pmatrix} \cos{}^{\theta}/_{2} - i\sin{}^{\theta}/_{2}\, \hat{n}_z & -i\sin{}^{\theta}/_{2}\, \hat{n}_x - \sin{}^{\theta}/_{2}\, \hat{n}_y \\ -i\sin{}^{\theta}/_{2}\, \hat{n}_x + \sin{}^{\theta}/_{2}\, \hat{n}_y & \cos{}^{\theta}/_{2} + i\sin{}^{\theta}/_{2}\, \hat{n}_z \end{pmatrix}$$

Rotation φ around z axis;

$$\begin{pmatrix} e^{-i\varphi/2} & 0 \\ 0 & e^{i\varphi/2} \end{pmatrix}$$

Rotation around z axis changes the relative phase of the two components of a spinor.

Rotation around y-axis:

$$\begin{pmatrix} \cos{}^{\theta}/_{2} & -\sin{}^{\theta}/_{2} \\ \sin{}^{\theta}/_{2} & \cos{}^{\theta}/_{2} \end{pmatrix}$$

Combination (Euler angles) corresponds to rotating the spinor pointing in +z-direction to the direction given by the spherical coordinates θ, ϕ:

$$\begin{pmatrix} e^{-i\varphi/2} & 0 \\ 0 & e^{i\varphi/2} \end{pmatrix} \begin{pmatrix} \cos{}^{\theta}/_{2} & -\sin{}^{\theta}/_{2} \\ \sin{}^{\theta}/_{2} & \cos{}^{\theta}/_{2} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} \cos{}^{\theta}/_{2}\, e^{-i\varphi/2} \\ \sin{}^{\theta}/_{2}\, e^{i\varphi/2} \end{pmatrix}$$

This is exactly the form of the **most general** state possible if we identify ${}^{\theta}/_{2} = \gamma$ and $\varphi = \Delta\delta$! This means that for each possible state of a spin-1/2 particle, there is (exactly) one direction in space (given by the spherical coordinates θ, ϕ) such that it is in the eigenstate with $m_s = +1/2$ of the spin operator pointing in that direction, $\hat{n}\cdot\vec{S}$. Of course, for all **other** directions (except $-\hat{n}$), the state is **not** in an eigenstate of the corresponding spin operator, and therefore will have a statistical uncertainty for any measurement of the spin component along that direction.

Force Due to a magnetic field B on a length *s* of wire carrying current *I*: *F = BIs*

Torque on square loop with side length *s* = $2 \cdot s/2\ BIs \sin\theta$

Magnetic moment = Is^2

$$\vec{\mu} = Ia\,\hat{n}$$

$$\vec{\tau} = \vec{\mu} \times \vec{B}$$

Work dW= $\tau d\theta$

Let initial orientation be at θ=90

Work done = $\mu B \int_{90}^{\theta_{final}} \sin\theta\, d\theta$

Potential energy stored in the loop, $V_{pot} = -\mu B \cos\theta_{final}$

Magnetic dipole moment of a single charge *q* orbiting at fixed radius *r* with velocity *v*:

$$\mu = \frac{qv}{2\pi r} r^2 = \frac{qvr}{2} = \frac{q}{2mc} L$$

Interaction Hamiltonian is given by,

$$H_{int} = -\vec{\mu}.B$$

$$= -\frac{q}{2mc}\vec{J}.B$$

Electron Spin : $H_{int} = -g\frac{q}{2mc}\vec{S}.\vec{B}$

$= g\frac{e\hbar}{2mc}\frac{1}{2}\vec{\sigma}.\vec{B}$ Here $\mu_B = \frac{e\hbar}{2mc}$ and $g = 2 \cdot (1.00116)$

$= -\gamma\vec{S}.\vec{B}$

In general Hamiltonian can have..,

$|\psi >_{spatial} \otimes \chi$; $\chi = \begin{pmatrix} a \\ b \end{pmatrix}$

$$H = \frac{\vec{P}^2}{2m}\mathbb{1} + g\mu_B \frac{1}{2}\vec{\sigma}.\vec{B}$$

$$\left|\psi >_{(t=0)} => \right|\psi >_{(t)} = e^{-\frac{iHt}{\hbar}}|\psi >_{(t=0)}$$

Rotation around axis of $\vec{B}$ by an angle of $g\mu_B \frac{Bt}{\hbar}$

$$|\psi >_{(t)} = e^{-ig\mu_B B\hat{b}.\frac{\vec{\sigma}}{2}\frac{t}{\hbar}}$$

Here $g_{proton} = 2(2.79)$

$g_{neutron} = 2(-1.91)$

If the magnetic field is inhomogeneous;

Force = $-\nabla\overrightarrow{V_{pot}} = \vec{\mu}.\nabla\vec{B}$

Consequence: Stern-Gerlach apparatus which can measure the angular momentum (spin) component along the z-direction determined by the field direction.