Angular Momentum

For any vector operator $\vec{V} = \{v_x, v_y, v_z\}$

$$[V_i, L_z] = i\hbar\epsilon_{ijk}V_k$$

For infinitesimal rotation,

$$[V_i, L_z]\delta\phi_j = i\hbar\epsilon_{ijk}V_k\delta\phi_j$$

For rotation about any axis,

$$[V_i, \vec{n}.L]d\phi_n = i\hbar(\vec{n} \times \vec{V})_i d\phi_n$$

We know,

$$[L^2, L_z] = 0$$

It means we can find the common set of eigen function for L^2 and L_z.
Suppose we have eigen function $|\alpha, m>$ such that,

$$L^2|\alpha, m>= \alpha|\alpha, m>$$

$$L_z|\alpha, m>= \hbar m|\alpha, m>$$

(assuming eigen values of L_z are integer multiple of $\hbar$.)
To find eigen values of L^2:
Lets invent,

$$L_+ = L_x + iL_y$$

$$L_- = L_x - iL_y$$

Also,$L_- = (L_+)^\dagger$
Now,

$$[L_z, L_\pm] = [L_z, L_x] \pm i[L_z, L_y]$$

$$= i\hbar L_y \pm i(i\hbar L_x)$$

$$= i\hbar L_y \pm \hbar L_x$$

$$= \pm\hbar L_\pm$$

Again,

$$L^2[L_+|\alpha, m>] = L_+L^2|\alpha, m>$$

$$= \alpha L_+|\alpha, m>$$

$$L_z[L_+|\alpha, m>] = [L_+L_z + [l_z, L_+]]|\alpha, m>$$

$$= m\hbar L_+|\alpha, m>] + \hbar L_+|\alpha, m>$$

$$= (m+1)\hbar L_+|\alpha, m>$$

So,

$$L_\pm|\alpha, m>= C_\pm(\alpha, m)|\alpha, m \pm 1>$$

C is introduced to account normalization.
In addition,m can't be arbitrary. It should be fixed within some limit.
Here,

$$<\alpha, m|L^2 - L_z^2|\alpha, m>$$

$$=<\alpha, m|L_x^2 + L_y^2|\alpha, m>$$

$$= |L_x|\alpha, m>|^2 + |L_y|\alpha, m>|^2 whichis > 0$$

Therefore,

$$<\alpha, m|L^2 - L_z^2|\alpha, m>= \alpha - m^2\hbar^2 > 0$$

The value of m is constrained by the this equation.m can't be arbitrarily positive or negative.
It should follow that,

$$L_+|l, m_{max}>= 0$$

Here,

$$L_-L_+ = L_x^2 + L_y^2 + i[L_x + L_y]$$

$$= L^2 - L_z^2 + i(i\hbar L_z)$$

$$= L^2 - L_z^2 - \hbar L_z$$

So,

$$L^2 - L_z^2 - \hbar L_z|l, m_{max}>= 0$$

$$or, (\alpha - \hbar^2 m^2 - \hbar^2 m)|l, m_{max}>= 0$$

Therefore,

$$\alpha = \hbar^2 m_{max}(m_{max} + 1)$$

$$= \hbar^2 l(l+1)$$

calling $m_{max} = l$
Thus we obtain,

$$L_z|l, m>= \hbar m|l, m>$$
$$L^2|l, m>= \hbar^2 l(l+1)|l, m>$$

This gives the condition for m_{max}.Similarly, we can find that,$m_{min} = -l$
Hence, the range for m is $-l$ to $+l$.
Now,
For,
$l = 0$, $m = 0$
$l = \frac{1}{2}$, $m = \frac{-1}{2}, \frac{1}{2}$
$l = 1$, $m = 1, 0, -1$
$l = \frac{3}{2}$, $m = \frac{-3}{2}, \frac{-1}{2}, \frac{1}{2}, \frac{3}{2}$
We have $2l + 1$ eigen vectors for any given l.
L_z should give only integer value.In order to generalize $\frac{1}{2}$ integer value we invent $\vec{J}$ which describes rotations such that J has eigen function $|j, m_j>$.
Now,

$$J^2|j, m_j>= \hbar^2 j(j+1)|j, m_j>$$

and,

$$J_z|j, m_j>= \hbar m_j$$
$$j = o, \frac{1}{2}, ...\frac{n}{2}; m_j = -j, -j+1, ...+j$$

Quantum mechanics tells us that there might be rotations of space that can't be defined by $\vec{r} \times \vec{p}$ which exist in classical mechanics.

Lets evaluate $C_\pm$:

$$|C_\pm|^2 = |L_+|l, m>|^2$$
$$=< l, m|L_- L_+|l, m>$$
$$= \hbar^2[l(l+1) - m^2 - m]$$
$$or, C_+ = \hbar\sqrt{l(l+1) - m(m+1)}$$

Similarly, $C_- = \hbar\sqrt{l(l+1) - m(m-1)}$

In spherical coordinates,

$$x = rsin\theta cos\phi, y = rsin\theta sin\phi, z = rcos\theta$$

And,

$$L_+ = \hbar e^{i\phi}\left(\frac{\partial}{\partial\theta} + \frac{icot\theta\partial}{\partial\phi}\right)$$

$$L_- = \hbar e^{-i\phi}\left(-\frac{\partial}{\partial\theta} + \frac{icot\theta\partial}{\partial\phi}\right)$$

and from HW Set 9:

$$\vec{L}^2 = -\hbar^2\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\partial\theta + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right).$$

We know that

$$L_+|l, l\rangle = 0$$

from which we can deduce that

$$Y_l^l(\theta,\phi) = (-1)^l\sqrt{\frac{(2l+1)!}{4\pi}}\frac{1}{2^l l!}\sin^l\theta e^{il\phi}.$$

All other Y_l^m's can be deduced from this by repeatedly applying L_-.

An arbitrary state with given angular momentum quantum numbers l, m can be written

$$\psi(r,\theta,\phi) = R(r)Y_l^m(\theta,\phi)$$

where for

$$m > 0, Y_l^m(\theta,\phi) = (-1)^m\sqrt{\frac{2l+1(l-m)!}{4\pi(l+m)!}}P_l^m(cos\theta)e^{im\phi}$$

and for

$$m < 0, Y_l^m(\theta,\phi) = (-1)^m\sqrt{\frac{2l+1(l-|m|)!}{4\pi(l+m)!}}P_l^{|m|}(cos\theta)e^{im\phi}$$

For a particle in a rotationally symmetric potential $V(r)$ the Hamiltonian in spherical coordinates is

$$H = \frac{-\hbar^2\nabla^2}{2m} + V(r)$$

with

$$\nabla^2 = \left(\frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial\theta} \sin\theta \partial\theta + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2}{\partial\phi^2} \right).$$

Therefore, we can write the Hamiltonian as

$$= \frac{L^2}{2mr^2} - \frac{\hbar^2 D_r}{2m} + V(r)$$

where we define

$$D_r = \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r}.$$

We are interested in eigenstates of the Hamiltonian

$$H\psi_{E,l,m} = E\psi_{E,l,m}$$

$$or, \frac{-\hbar^2 D_r \psi}{2m} + \frac{L^2 \psi}{2mr^2} + V(r) = E\psi$$

Assume $\psi = R(r) Y_l^m(\theta, \phi)$. Also

$$\frac{1}{2mr^2} L^2 Y_l^m = \frac{\hbar^2}{2mr^2} l(l+1) Y_l^m(\theta, \phi)$$

The radial equation is therefore,

$$-\frac{\hbar^2}{2m} D_r R(r) + \left[\frac{\hbar^2 l(l+1)}{2mr^2} + V(r) \right] R(r) = ER(r)$$

Let $R(r) = \frac{U(r)}{r}$
Then,

$$\left[\frac{d^2}{dr^2} - \frac{l(l+1)}{r^2} - \frac{2m}{\hbar^2} [V(r) - E] \right] U(r) = 0.$$

For now, let's restrict ourselves to the case $V(r) = 0$ (it could be an arbitrary constant, but we can always set it to zero by redefining E). In this case, the Schrödinger equation reads

$$\left[\frac{d^2}{dr^2} - \frac{l(l+1)}{r^2} - k^2 \right] U(r) = 0$$

with

$$k^2 = \frac{2m}{\hbar^2} [V - E].$$

Now lets look at the limiting cases:

When $r \to 0$, for $l = 0, U = \begin{cases} sinkr \\ coskr \end{cases}$

We can ignore the 2nd possibility since it would lead to an overall wave function

$$\psi = R(r)Y_0^0(\theta, \phi) \propto \frac{1}{r}$$

at the origin but we know that the Laplacian of this is a delta function at the origin. So it cannot solve the Schrödinger equation unless the potential itself is a delta function.

For $l \neq 0$,

$$\frac{l(l+1)}{r^2}$$

dominates over the potential term, so

$$U'' - \frac{l(l+1)}{r^2}U = 0.$$

Its solutions are,

$$U \approx r^{l+1}$$
$$U \approx \frac{1}{r^l}$$

Again, the 2nd solution cannot be correct at the origin since the full wave function

$$\psi = R(r)Y_l^m(\theta, \phi)$$

could not be normalized:

$$|\psi|^2 = \int r^2 dr \int d\cos\theta \int d\phi |R(r)|^2 |Y_l^m(\theta, \phi)|^2$$
$$= \int r^2 dr |R(r)|^2 \approx \int r^2 dr \frac{1}{r^{2l+2}}$$

which explodes at the origin.

We already know what the solution for a free particle looks like in cartesian coordinates:

$$\psi(\vec{r}) = \frac{1}{(2\pi\hbar)^{3/2}} e^{\frac{i\vec{p}\cdot\vec{r}}{\hbar}}$$

To solve the radial equation in spherical coordinates,

$$U'' = \frac{l(l+1)U}{r^2} - k^2 U,$$

we define $\rho = kr$ so that $U(r) = V(kr) = V(\rho)$. Then,

$$U'' = k^2 v'' = \frac{l(l+1)k^2}{\rho^2} U - U$$

or

$$\frac{d^2v}{d\rho^2} + \left(1 - \frac{l(l+1)}{\rho^2}\right) v = 0.$$

Introduce $d_l = \frac{\partial}{\partial\rho} + \frac{l+1}{\rho}$ and $d_l^\dagger = \frac{-\partial}{\partial\rho} + \frac{l+1}{\rho}$. This yields

$$d_l d_l^\dagger = \frac{-\partial^2}{\partial\rho^2} + \frac{l(l+1)}{\rho^2}$$

$$d_l^\dagger d_l = \frac{-\partial^2}{\partial\rho^2} + \frac{(l+1)(l+2)}{\rho^2}$$

Now we can rewrite the equation for v:

$$d_l d_l^\dagger v_l(\rho) = v_l(\rho)$$

Multiplying from the left with $d_l^\dagger$ and regrouping the ladder operators we find

$$d_l^\dagger d_l d_l^\dagger v_l(\rho) = \left(\frac{-\partial^2}{\partial\rho^2} + \frac{(l+1)(l+2)}{\rho^2}\right) d_l^\dagger v_l(\rho) = d_l^\dagger v_l(\rho)$$

which is the equation for $v_{l+1}(\rho)$. We can therefore conclude that the ladder operator $d_l^\dagger$ acting on $v_l(\rho)$ turns it into (a multiple of) the radial eigenstate for the next higher l. Thus, if we can start with the solution for $l = 0$, we can produce all higher-l solutions by repeated application of the ladder operator.

We already know that for $l = 0$ the solutions are

$$v_0(\rho) = \begin{cases} sin\rho \\ -cos\rho \end{cases}$$

If we want to find $v_l(\rho)$ then,we have to carry out the following:

$$v_l(\rho) = d_{l-1}^\dagger d_{l-2}^\dagger ......d_0^\dagger v_0(\rho).$$

We can rewrite this if we remind ourselves that we are really after the radial solutons $R(r) = U(r)/r = v(\rho)/r$:

$$R_{E,l=0}(r) = \begin{cases} \frac{sinkr}{r} = \frac{ksinkr}{kr} = kj_0(kr) \\ \frac{-coskr}{r} = \frac{-kcoskr}{kr} = -kn_0(kr) \end{cases}$$

In general

$$R_{E,l}(r) = \frac{v_l(\rho)}{r} = k \begin{cases} j_l(\rho) \\ n_l(rho) \end{cases}$$

where j_l is the spherical Bessel function and n_l is the Neumann function.

We can show (using the expression involving the ladder operator) that

$$j_l(\rho) = (-\rho)^l \left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\right)^l j_0(\rho)$$

$$n_l(\rho) = (-\rho)^l \left(\frac{1}{\rho} \frac{\partial}{\partial \rho} \right)^l n_0(\rho)$$

While we know that only the spherical Bessel functions can contribute at the origin, this more general solution is useful if we have to "piece together" a solution for a potential that has one value near the origin and a different value for $r > a$ (particle in a "spherical box" of radius a). In that case, the j_l contribute in the center and we have to use the continuity of the wave function and its derivative to match to a combination of Bessel and Neumann functions at $r \geq a$.

For a truly free particle, any value of $E > 0$ is allowed and there is exactly one solution for each combination E, l, m. On the other hand, if the particle is locked inside a rigid sphere ($V(r) = \infty$ for $r > a$), we have to require that $R_{E,l}(r) = 0$ for $r = a$. This leads to only certain values of k and therefore E being permissible – we once again get quantized energy eigenstates. For instance, for $l = 0$, we have to require $ka = n\pi$ with integer n.

Finally, we can make the connection with the cartesian form of the free particle wave function - since the eigenstates in spherical coordinates must form a complete basis, we should be able to express the plane wave as a linear combination of solutions in spherical coordinates. For simplicity, let's assume that the wave travels along the z-direction, $\vec{p} = p\hat{z}$. Then

$$\psi(\vec{r}) = \frac{1}{(2\pi\hbar)^{3/2}} e^{\frac{ipz}{\hbar}} = \frac{1}{(2\pi\hbar)^{3/2}} e^{\frac{ipr\cos\theta}{\hbar}}$$

which is already expressed in spherical coordinates. Since there is no ϕ dependence, we can assume that only Y_l^0's can contribute. Indeed, the plane wave can be written as

$$\psi(\vec{r}) = \Sigma_l c_l k j_l(kr) Y_l^0(\theta, \phi).$$