Graduate Quantum II
Multi-Fermion Symmetry
Spring 2013

1 Two Particle State

Bose-Einstein statistics: completely symmetric under the exchange of two particles
Fermi-Dirac statistics: $(-)$ minus sign under exchange of any two particles

Subset of possible two-particle states: Assume each particle is in a well-defined 1 particle state:

$$|\Psi_1\rangle \in \mathbf{V}$$
$$|\Psi_2\rangle \in \mathbf{V}$$
$$|\Psi\rangle = |\Psi_1\rangle \otimes |\Psi_2\rangle$$

The ordering tells us which particle in which state (1st is 1st, 2nd in 2nd). For two boson-states, we need to use the symmetric superposition with the (+), and for two-Fermion states, we need the anti-symmetric superposition with the (-) as below:

$$\frac{1}{\sqrt{2}}(|\Psi_1\rangle|\Psi_2\rangle \pm |\Psi_2\rangle|\Psi_1\rangle)$$

If the one-particle states are properly normalized and orthogonal, these two-particle states are in general "automatically" normalized, as well. For the symmetric case, this only works if $|\Psi_1\rangle \neq |\Psi_2\rangle$; otherwise simply use the simple product $|\Psi_1\rangle|\Psi_1\rangle$.

2 Two non-interacting Particles

As an example, let's study again the harmonic oscillator. Each particle is independently moving in the (one-dimensional) quadratic potential well. We will treat both symmetric and anti-symmetric wave functions simultaneously. Symmetric wave function could mean 2 fermions with anti-aligned spin states (total spin $S = 0$) and symmetric spatial wave function, or two spin-0 bosons. Anti-symmetric wave functions would be needed for two fermions coupling to total spin $S = 1\hbar$.

Single-particle states in this case fulfill

$$H|\Psi_n\rangle = E|\Psi_n\rangle$$

where $E = (n + \frac{1}{2})\hbar\omega$.

Possible two-particle states:

$$\frac{1}{\sqrt{2}}(|\Psi_n\rangle|\Psi_m\rangle \pm |\Psi_m\rangle|\Psi_n\rangle)$$

To find the probability of one particle at one place (at x_1) and the other at x_2, we use the projection operator:

$$\frac{1}{\sqrt{2}}(\langle x_1 x_2| \pm \langle x_2 x_1|)$$

$$\frac{1}{\sqrt{2}}(\langle x_1 x_2| \pm \langle x_2 x_1|)\frac{1}{\sqrt{2}}(|\Psi_n\rangle|\Psi_m\rangle \pm |\Psi_m\rangle|\Psi_n\rangle) = \frac{2}{2}(\Psi_n(x_1)\Psi_m(x_2) \pm \Psi_n(x_2)\Psi_m(x_1))$$

What is the probability, $dProb(x_1, x_2, dx_1, dx_2)$? Can't ask which particle is which. If particles don't see each other/don't interact:

$$= \left[|\Psi_n(x_1)|^2|\Psi_m(x_2)|^2 + |\Psi_n(x_2)|^2|\Psi_m(x_1)|^2 \pm 2\Re(\Psi_n^*(x_1)\Psi_m^*(x_2)\Psi_n(x_2)\Psi_m(x_1))\right] dx_1 dx_2.$$

The first two terms is what one expects from "classical" physics: since we don't know which particle is in which state, the probabilities have to add. However, the last term generates interference effects that would not exist classically. For instance, for the symmetric (+) case, the probability to find both particles at the *same* spot increases to nearly double the classical value (simple addition of probabilities) - this is equivalent to bright interference fringes in light going through a double slit. Vice versa, the probability to find two particles governed by the asymmetric wave function in the same spot is zero! This is the basis of the statement that "bosons like to congregate" and "fermions like to stay apart".

BTW, this equation also explains why, in practical terms, we do not have to worry about every fermion in the Universe being entangled (via antisymmetrization) with every other fermion of the same type (e.g., an electron on Earth with an electron on Mars): If the two wave functions have non-overlapping "support" (i.e., the x-range over which $|\Psi_n(x)$ is non-zero does not – or *nearly* not – overlap with the corresponding range over which $|\Psi_m(x) \neq 0$), then only the first or second term can be non-zero for any pair x_1, x_2 and neither the other square term nor the interference term matter.

3 Identical particles that do interact with each other

Again we consider the Harmonic Oscillator, but this time the two particles are connected to *each other* by a spring, instead of moving independently in the same "fixed" potential.

$$\frac{P^2}{2M} + \frac{p^2}{2\mu} + \frac{1}{2}\omega^2 \mu x$$

Change of basis: $|x_1 x_2\rangle \to |xX\rangle$. For $m_1 = m_2$:

$$X = \frac{x_1 + x_2}{2}$$
$$x = x_1 - x_2$$

What functions of x and X fulfill the requirement to be anti-symmetric or symmetric? By definition of X, $\Psi_{cm}(X)$ is always symmetric under exchange $1 \leftrightarrow 2$. So the determination is left to $\Psi_{rel}(x)$. Since under $1 \leftrightarrow 2$, $x \to -x$, we have for symmetric wave functions that Ψ_{rel} must be "even", $\Psi_{rel}(-x) = \Psi_{rel}(x)$, and for anti-symmetric functions Ψ_{rel} must be "odd": $\Psi_{rel}(-x) = -\Psi_{rel}(x)$.

The even and odd properties relate to the behavior under the parity operation of the corresponding one-particle wave functions (reversal of all signs of all components of position and momentum vectors). For rotationally symmetric Hamiltonian,

$$\Psi_{rel}(\vec{r}) = R_{nl}(r)Y_l^m(\theta, \phi)$$

Under parity:

$$r \to r, \theta \to \pi - \theta, \phi \to \phi + \pi \Rightarrow \Psi_{rel} \to \Psi_{rel}(-\vec{r}) = R_{nl}(r) Y_l^m(\pi - \theta, \phi + \pi)$$

As an example, we study the case $l = m$:

$$Y_l^l = \sin\theta^l e^{il\phi} \to \sin(\pi - \theta)^l e^{il\phi} e^{il\pi}$$

where $e^{il\pi} = (-1)^l$ is even if l is even and odd if l is odd. This is also true for all other values of m and therefore l determines if the wave function is symmetric or anti-symmetric in space.
For example, 2 spin $\frac{1}{2}$ fermions: if S=0, spin anti-symmetry, spatially symmetric$\to$ $l = 0, 2, 4, ...$ If S=1, spin symmetry, spatially anti-symmetric$\to$ $l = 1, 3, 5, ...$

4 Generalize to N Particles

States with N particles, assume N different single-particle wave functions: $|\Psi_a\rangle, |\Psi_b\rangle, \ldots |\Psi_{lN}\rangle$. Then the symmetric wave function for bosons would be

$$\frac{1}{\sqrt{N!}} \sum_{allpossiblepermutations} |\Psi_a\rangle \otimes |\Psi_b\rangle \otimes |\Psi_{lN}\rangle$$

For Fermions, each permutation must be multiplied by its "sign" (plus for an even number of "swaps", minus for an odd number):

$$\frac{1}{\sqrt{N!}} \sum_{permutations} (sign) |\ldots\rangle \ldots |\ldots\rangle$$

No ambiguity to write down as the Slater determinant of $N \times N$ matrix:

$$\frac{1}{\sqrt{N!}} \begin{bmatrix} \Psi_a & \Psi_b & \ldots & \Psi_{lN} \\ \Psi_a & \Psi_b & \ldots & \Psi_{lN} \\ \Psi_a & \Psi_b & \ldots & \Psi_{lN} \end{bmatrix}$$

This is the short way to get the wave function for n-fermion state, although the normalization may have to be determined by hand unless all single-particle wave functions are orthonormal to each other.
Example 1: ^{4}He atom has 2 e^-. We know ground state is Ψ_{100} and the two possibilities are $\uparrow$ or $\downarrow$. We take the determinant of the matrix below:

$$\frac{1}{\sqrt{2!}} \begin{bmatrix} \Psi_{100}\uparrow & \Psi_{100}\downarrow \\ \Psi_{100}\uparrow & \Psi_{100}\downarrow \end{bmatrix}$$

This matrix yields the determinant as:

$$\frac{1}{\sqrt{2}} |\Psi_{100}\rangle |\Psi_{100}\rangle (\uparrow\downarrow - \downarrow\uparrow)$$

In this case, we have an anti-symmetric spin state and a symmetric spatial state.
Example 2: Particles in 1D box:
For n^{th} state, wave number $k_1 = n\pi/L$, momentum $p_1 = n\hbar\pi/L$. Total volume in phase space for each state is $V_x \times V_p = \hbar$. Since each state can be "filled" with two fermions (1 spin up, one spin down), the total volume in phase space occupied by N fermions is $\frac{N\hbar\pi}{2}$. Packing more fermions into the space requires more kinetic energy (higher and higher momenta) $\to$ Fermi gas.