Additional Literature

Introduction to Quantum Mechanics, David J. Griffiths (Pearson) (Undergrad text)

Dance of the Photons, Anton Zeilinger (Farrar, Straus and Giroux) (Theoretically a "General Audience" book)

Quantum Mechanics" by D.H. McIntyre (Pearson) (Undergrad text)

Spin-1/2 states:

Each individual particle is described by a vector in $\mathbb{C}^2$. Basis:

$|\uparrow>=\begin{pmatrix}1\\0\end{pmatrix}$, $|\downarrow>=\begin{pmatrix}0\\1\end{pmatrix}$

For the most general state vector of *one* particle state, 2 real parameters required, as opposed to 2 complex parameters

Most general state (single particle): $\begin{pmatrix}\cos\frac{\theta}{2}e^{i\phi/2}\\ \sin\frac{\theta}{2}e^{-i\phi/2}\end{pmatrix}$

This state describes a particle with its spin pointing in the direction of the unit vector given by the polar angles θ and ϕ, $\hat{n}(\theta) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$.

From now on, restrict ourselves to $\phi = 0$, i.e., spin vectors in the $x - z$ plane. The vector spin operator is

$\vec{S}_{1/2} = \frac{\hbar}{2}\vec{\sigma}$. It is easy to show that

$$\vec{S}_{1/2}\cdot\hat{n}(\theta)\begin{pmatrix}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}\end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}\end{pmatrix}$$

$$\vec{S}_{1/2}\cdot\hat{n}(\theta)\begin{pmatrix}-\sin\frac{\theta}{2}\\ \cos\frac{\theta}{2}\end{pmatrix} = -\frac{\hbar}{2}\begin{pmatrix}-\sin\frac{\theta}{2}\\ \cos\frac{\theta}{2}\end{pmatrix}$$

Therefore we can say that the two states

$$\nearrow=\begin{pmatrix}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}\end{pmatrix}$$

$$\swarrow=\begin{pmatrix}-\sin\frac{\theta}{2}\\ \cos\frac{\theta}{2}\end{pmatrix}$$

are eigenstates to the spin projection in $\hat{n}(\theta)$ direction with eigenvalues $\pm\frac{\hbar}{2}$. If a particle is prepared in spin state $|\uparrow>$ and then this spin projection along $\hat{n}(\theta)$ is measured, the probability to measure $+\frac{\hbar}{2}$ is $\cos^2\frac{\theta}{2}$ and the probability to measure $-\frac{\hbar}{2}$ is $\sin^2\frac{\theta}{2}$. In general, if the particle has initially spin "up" along some direction (in the x-z plane) and then the spin is measured along some other axis in the same plane that forms an angle θ_{rel} with the first one, the probabilities for "up" and "down" are $\cos^2 \frac{\theta_{rel}}{2}$ and $\sin^2 \frac{\theta_{rel}}{2}$.

Two-particle states:

Two spin-1/2 particles

Two-particle state vector space described by $\mathbb{C}^2 \otimes \mathbb{C}^2 = \mathbb{C}^4$

Possible basis: $\uparrow\uparrow, \uparrow\downarrow, \downarrow\uparrow, \downarrow\downarrow$

$S_{z_1} = S_{z_1} \otimes \mathbb{I}_2$: acting on the first particle only

$S_{z_{tot}} = S_{z_1} + S_{z_2} \rightarrow \vec{S} = \vec{S}_1 + \vec{S}_2$

$S^2 = \vec{S}^2$

3 alternative basis states: $\uparrow\uparrow, \frac{1}{\sqrt{2}}(\uparrow\downarrow + \downarrow\uparrow), \downarrow\downarrow$ with $S^2 = 1, M_S = 1, 0, -1$

4th basis state with $S^2 = 0, M_S = 0$:

$\frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)$

So far, have defined this spin=0 state in terms of eigenstates of $\mathbf{S}_z$. However, all directions in space are equivalent $\rightarrow$ lets see what we get if we use an arbitrary spin direction in the x-z plane (with corresponding new basis states).

$$\nearrow = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix}$$

$$\swarrow = \begin{pmatrix} -\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \end{pmatrix}$$

Alternative: $\frac{1}{\sqrt{2}}(\nearrow\swarrow - \swarrow\nearrow)$

$= \frac{1}{\sqrt{2}}((\cos\frac{\theta}{2} \uparrow + \sin\frac{\theta}{2} \downarrow) \otimes (-\sin\frac{\theta}{2} \uparrow + \cos\frac{\theta}{2} \downarrow) - (-\sin\frac{\theta}{2} \uparrow + \cos\frac{\theta}{2} \downarrow$
$) \otimes (\cos\frac{\theta}{2} \uparrow + \sin\frac{\theta}{2} \downarrow))$

$= \frac{1}{\sqrt{2}}(-\cos\frac{\theta}{2}\sin\frac{\theta}{2} \uparrow\uparrow + \cos^2\frac{\theta}{2} \uparrow\downarrow - \sin^2\frac{\theta}{2} \downarrow\uparrow + \cos\frac{\theta}{2}\sin\frac{\theta}{2} \downarrow\downarrow + \sin\frac{\theta}{2}\cos\frac{\theta}{2} \uparrow\uparrow$
$+ \sin^2\frac{\theta}{2} \uparrow\downarrow - \cos^2\frac{\theta}{2} \downarrow\uparrow - \sin\frac{\theta}{2}\cos\frac{\theta}{2} \downarrow\downarrow)$

$= \frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)$

Conclusion: it doesn't matter which direction in space we choose for the original basis - all lead to the same spin-0 state (which makes sense since spin-0 means that the state is invariant under all rotations). This state is an "entangled" state as we will now show.

Entanglement:

Example: decay of $\sigma^0 \to e^+ e^-$

Spin-0 particle decays into two spin-1/2 particles

Two-particle wave function that comes out of this decay must take the original spin-0 into account - the total spin of the 2-particle system is still 0.

$\frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)$: this system is entangled (cannot be written as the simply product of just two single-particle wave functions); you *must* write the wavefunction as a linear combination of two or more simple products of wavefunctions

Can make measurements on this system of total spin, total spin projected along the z-axis. Can also make similar measurements on only one of the particles, ie, $\hat{n}(\theta) \cdot \vec{S}_1$

Can measure either $\frac{\hbar}{2}$ or $-\frac{\hbar}{2}$

It's possible to construct a projection operator, ie, state of system after measurement

If you measure $\hat{n}(\theta) \cdot \vec{S}_1$ and find $\frac{\hbar}{2}$: wave function is projected onto the subspace where particle 1 has this particular spin orientation, with projection operator$\nearrow$, $\mathbb{P}_{\mathbb{V}_{\frac{\hbar}{2}\hat{n}(\theta)}} = |\nearrow\swarrow><\nearrow\swarrow| + |\nearrow\nearrow><\nearrow\nearrow|$ (Second term in projection operator doesn't actually contribute.) Collapsed wavefunction: $|\nearrow\swarrow>$

If result is $-\frac{\hbar}{2}$: $\swarrow$, collapsed wavefunction is $|\swarrow\nearrow>$.

Each result has probability 1/2.

Note:

Second particle must be in the opposite state measured for the first particle, along the same direction $\hat{n}(\theta)$, no matter what direction the first measurement (Alice) chose - so both the direction and the result immediately change the wave function of the second particle.

Measurement of first particle instantaneously collapses the wavefunction of the second particle $\to$ "spooky (instantaneous) action at a distance".

EPR: Einstein's objections to quantum mechanics (entanglement), especially in light of special relativity

Proposed alternative: missing information (hidden variable)

Thought Experiment:

Assume A makes the first measurement

A and B measure same angle $\rightarrow$ 100% (anti-)correlation; whenever A measures up, B measures down and vice versa.

A and B measure different angle $\rightarrow$ probabilites of second measurement results can be determined by results of first measurement

A measures in direction $\uparrow$, B measures in direction $\nearrow$

If A measures $\frac{\hbar}{2} \rightarrow\uparrow$, Probability of B measuring "up": $\sin^2\frac{\theta}{2}$, Probability of B measuring "down": $\cos^2\frac{\theta}{2}$

In general, whenever A and B measure along two different axes which enclose an angle θ between them, and if A measures "up", then the probability of B measuring 'up": $\sin^2\frac{\theta}{2}$, Probability of B measuring "down": $\cos^2\frac{\theta}{2}$ (since the situation should be rotationally symmetric).

Bell's theorem (Local hidden variable theory):

Assume Einstein is correct, particles must "know" from the beginning what the outcome of any possible measurement should be. Can think of each particle carrying a 'hidden list" containing information about all possible measurement outcomes. Must cover *all* possibilities since the direction chosen by A or B may be chosen so late that the information cannot be transferred back to the other particle in time (with light speed or less).

Each particle will have its own table to tell it what to do when we make a measurement. However, the two lists must be 100% correlated for any given direction, i.e. if I only knew what particle 1 has as its entry for a specific direction (up or down), I would also know with 100% certainty what the other particle's entry for *that* specific direction must be (namely, the opposite); otherwise, we couldn't guarantee the 100% correlation observed when both A and B measure along the same direction.

We don't know about particle states before measurement, but the particles do! The lists may change from one instance to the next (if we keep repeating the experiment with a number of decays) in an unknown way, but at least there must be a well-defined probability for each list to have a specific set of entries. We will look specifically at measurements along 3 specific directions:

- $\hat{n}_-$ points to the left of the z-axis, with an angle $\theta = -60°$
- $\hat{n}_0 = \hat{z}$ points along the z-axis, with an angle $\theta = 0°$
- $\hat{n}_+$ points to the right of the z-axis, with an angle $\theta = +60°$

Bell's theorem asks: What is the probability that A will measure "up" along $\hat{n}_{-1}$ and B measures "down" along $\hat{n}_0$? (ie, $\text{Prob}(A_{up}\hat{n}_-, B_{dn}\hat{n}_0)$)
Answer (according to quantum mechanics): Probability for A measuring "up" is always 1/2, and since the angle between the two axes is $\theta = 60°$ in this case, the probability of B measuring "down" if A measures "up" is given by $\cos^2 \frac{60°}{2} = \cos^2 30° = \frac{3}{4}$. The product of these two probabilities gives $\frac{3}{8}$.
Let's see instead what classical probability theory would predict if we assume the existence of the "hidden lists":
Since the two "lists" of the two particles must be 100% anti-correlated, we can say that equivalently $Prob(A_{up}\hat{n}_-, B_{dn}\hat{n}_0) = Prob(A_{up}\hat{n}_-, A_{up}\hat{n}_0) = Prob(A_{up}\hat{n}_-, A_{up}\hat{n}_0, A_{up}\hat{n}_+) + Prob(A_{up}\hat{n}_-, A_{up}\hat{n}_0, A_{dn}\hat{n}_+)$.
The last line simply lists two mutually exclusive but exhaustive possibilities that together make up all possible "lists" with entries $A_{up}\hat{n}_-$ and $A_{up}\hat{n}_0$. Therefore the sum of these two probabilities must add up to the probability for $A_{up}\hat{n}_-, A_{up}\hat{n}_0$ (just like the probabilities for "heads" and "tails" must add up to 1).
The first of these 2 terms must be $\leq Prob(A_{up}\hat{n}_-, A_{up}\hat{n}_+)$ since we are *relaxing* one condition on the list (by dropping the requirement $A_{up}\hat{n}_0$). In other words, if I toss a coin three times, the probability for any specific outcome for the first and last coin toss is at least as large as the probability for requiring the same two outcomes *plus* a certain outcome for the middle one.

For the same reason, the second term $\leq Prob(A_{up}\hat{n}_0, A_{dn}\hat{n}_+)$ (again, relaxing one requirement). This leads to Bell's inequality

$$\begin{aligned} Prob(A_{up}\hat{n}_-, B_{dn}\hat{n}_0) &\leq Prob(A_{up}\hat{n}_-, A_{up}\hat{n}_+) + Prob(A_{up}\hat{n}_0, A_{dn}\hat{n}_+) \quad (1) \\ &= Prob(A_{up}\hat{n}_-, B_{dn}\hat{n}_+) + Prob(A_{up}\hat{n}_0, B_{up}\hat{n}_+) \quad (2) \end{aligned}$$

L.h.s. $= \frac{3}{8}$ according to quantum mechanics (see previous page).
Each of the two terms on the r.h.s. yields $\frac{1}{8}$, also according to quantum mechanics: The measurement results for A has probability 1/2 (no matter what it is), and

$$Prob(A_{up}\hat{n}_-, B_{dn}\hat{n}_+) = \frac{1}{2}\cos^2\frac{120}{2} = \frac{1}{2}\frac{1}{4} = \frac{1}{8} \quad (3)$$

as well as

$$Prob(A_{up}\hat{n}_0, B_{up}\hat{n}_+) = \frac{1}{2}\sin^2\frac{60}{2} = \frac{1}{2}\frac{1}{4} = \frac{1}{8} \quad (4)$$

as discussed before (note that in the first case, the two axes enclose an angle of 120° and in the second case, they enclose 60°). Therefore, Quantum Mechanics predicts that the l.h.s. is *larger* than the r.h.s. ($\frac{3}{8} > \frac{1}{8} + \frac{1}{8}$) in direct violation of our derivation of Bell's inequality.
So, we must conclude that quantum mechanics contradicts Bell's inequality. But measurement shows that Bell's inequality is wrong and quantum mechanics is right. So the assumptions that went into this inequality (local hidden variables, classical probability theory) are violated by nature. This rules out that there are any "lists" of possible measurement outcomes that tell the particles "locally" what to do when encountering a measuring device (i.e., no "local hidden variables" are allowed).
The only good news: There is no super-luminal message sent between the particles - no actual information is transmitted in the collapse of the wave function. This can be seen in the fact that both particles and both observers enter this discussion in a completely symmetric fashion - I would get exactly the same results whether I assume A's measurement is first (and collapses the wave function), or B's measurement.