Lecture Notes: 22/01/2013

**Density Matrix**:

If $N_i$ particles are in the state $|\psi_i\rangle$ then the density matrix for such mixed state is,

$$\hat{\rho} = \sum p_i |\psi_i\rangle\langle\psi_i|$$

and $p_i = \frac{N_i}{N}$, $\sum_i p_i = 1$

The polarization is defined as,
$P_j = \sum_i p_i \langle\psi_i|\sigma_j|\psi_i\rangle$, $j = 0, 1, 2, 3$. We always have $P_0 = 1$ and $0 \leq |\vec{P}| \leq 1$.
If $\hat{O}$ is any observable then average of expectation value of $\hat{O}$ is,

$$\langle\bar{O}\rangle = \frac{1}{2}\sum_j tr(\hat{O}\sigma_j)P_j$$

Again,$\{\frac{1}{\sqrt{2}}\sigma_j\}$ forms a basis for operators. We can write $\hat{\rho}$ as,

$$\hat{\rho} = \frac{1}{2}\sum_j tr\left(\rho\sigma_j\right)\sigma_j$$

Also,

$$tr\left(\rho\sigma_j\right) = \sum_k \langle k| \sum_i p_i |\psi_i\rangle\langle\psi_i|\sigma_j|k\rangle$$

$$= \sum_i p_i \langle\psi_i|\sigma_j| \left(\sum_k |k\rangle k\langle k|\right)|\psi_i\rangle$$

$$= \sum_i p_i \langle\psi_i|\sigma_j|\psi_i\rangle$$

$$= P_j$$

Therefore,$\hat{\rho} = \frac{1}{2}\sum_j P_j\sigma_j$

**Multi Particle System**:

If we want to describe N particles then we need $V^{R^{3N}}$ dimensional vector space. the most general basis for this space will be,
$|x_1, y_1, z_1, x_2, y_2, z_2, ....x_N, y_N, z_N\rangle$
the wave function will look like,
$\int ... \int dx_1 dx_2 ....dx_N dy_1 ....dy_N dz_1 ....dz_N \psi(x_1, y_1, z_1 .......x_N, y_N, z_N)|x_1, y_1, z_1 .......x_N, y_N, z_N\rangle$

But there is always possibility that physical condition may reduce the dimension of this vector space.

**Operators**:

In case of two spin half particles, for instance, we can introduce z-component of spin of 1st particle as $S_{z1} = S_z \otimes 1$. This acts only on the first basis. Similarly, we can introduce operator for second particle which will be $S_{z2} = 1 \otimes S_z$.

In the same way, consider a three dimensional N particle vector space. If we let all particle move around and ask: what is x-component of position of $2^{nd}$ particle (not caring anything else)?
Ans: Obviously the answer has to be a real number because it has to be somewhere. $x_2 = -\infty .... + \infty$. The corresponding operator that describes this observable is $X_2 = \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1} \otimes X \otimes \mathbf{1} \otimes ... \otimes \mathbf{1}$.

To calculate the probability we have to find a projection operator,which would be,
$\hat{P_{x2}} = \int ... \int dx_1 dx_3 .... dx_N dy_1 .... dy_N dz_1 .... dz_N . |x_1, y_1, z_1 .... x_N, y_n, z_N\rangle \langle x_1, y_1, z_1 .... x_N, y_n, z_N|$
$x_2$ is fixed and integration is carried over all other variables.
The probability to measure $x_2$ is ,

$$\delta P(x_2, x_2 + \delta x) = \left[ \langle \psi | \hat{P_{x2}} | \psi \rangle \right] \delta x =$$

$$\left[ \int ... \int dx_1 dx_3 .... dx_N dy_1 .... dy_N dz_1 .... dz_N \psi^*(x_1, y_1, z_1 ....... x_N, y_N, z_N) \psi(x_1, y_1, z_1 ....... x_N, y_N, z_N) \right] \delta x$$

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**Time Evolution**:

The time evolution is governed by Schrodinger Equation,

$$i\hbar \frac{\partial |\psi\rangle}{\partial t} = H|\psi\rangle$$

Here H should be expressed in terms of all degrees of freedom.

**Two Particle Hilbert Space**:

Each particle is described by the vector space C. Two particle live in $C_1 \otimes C_2$ dimensional vector space.
We can take $(|k_1\rangle \otimes |k_2\rangle)$ a basis for $C_1 \otimes C_2$.Then the most general state will

be,

$$|\psi\rangle = \sum_{k_1}\sum_{k2} C_{k1}C_{k2}|k_1\rangle \otimes |k_2\rangle)$$

If we take two spin $\frac{1}{2}$ particles.Then,
Basis for $C_1$, $|k_1\rangle = \{|\uparrow\rangle_1, |\downarrow\rangle_1\}$
Basis for$C_2$, $|k_2\rangle = \{|\uparrow\rangle_2, |\downarrow\rangle_2\}$
The most general state in this vector space will be,

$$|\psi\rangle = \sum_{k_1}\sum_{k2} C_{k1}C_{k2}|k_1\rangle \otimes |k_2\rangle)$$

If $S = 0$ then

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle_1 \otimes |\downarrow\rangle_2 - |\downarrow\rangle_1 \otimes |\uparrow\rangle_2)$$

It is not eigenstate for $S_{z1}$ and $S_{z2}$ as well but it is eigenstate for $S_{ztotal} = S_{z1} + S_{z2}$

**Case of two Non-Interacting Particles**:

The Hamiltonian can be defined as,

$$H = \frac{{p_1}^2}{2m_1} + V_1(x_1) + \frac{{p_2}^2}{2m_2} + V_1(x_2)$$

We want to solve for $H|\psi\rangle = E|\psi\rangle$
In position space, we can write,

$$\left[\frac{-\hbar^2}{2m_1}\frac{\partial^2}{\partial x_1^2} + V_1(x_1, x_2)\right]\psi(x_1, x_2) + \left[\frac{-\hbar^2}{2m_2}\frac{\partial^2}{\partial x_2^2} + V_2(x_2)\right]\psi(x_1, x_2) = E\psi(x_1, x_2)$$

In case of non-interacting particles, we can write $\psi(x_1, x_2) = \psi(x_1)\psi(x_2)$
With the assumption that

$$H_1\psi(x_1) = E_1\psi(x_1)$$
$$H_2\psi(x_2) = E_2\psi(x_2)$$

Then we will have,

$$E_1\psi(x_1)\psi(x_1) + \psi(x_1)E_2\psi(x_1) = E\psi(x_1)\psi(x_2); E = E_1 + E_2.$$

Now, method of separation of variable could be used to solve this equation.

**Case of two interacting particle**:

The Hamiltonian can be defined as,

$$H = \frac{{p_1}^2}{2m_1} + \frac{{p_2}^2}{2m_2} + V(x_1 - x_2)$$

In this case method of separation of variable will not work so we employ a trick to get rid of this difficulty. We describe the system in new coordinates , $X = X_{cm} = (m_1 x_1 + m_2 x_2)/M$ and $x = x_{rel} = x_1 - x_2$; then we can write our hamiltonian as

$$H = \frac{p_{total}^2}{2M} + \frac{p_{rel}^2}{2\mu} + V(x_{rel})$$

In such case the wave function can be written as,

$$\psi = \psi_{cm}(X_{cm})\psi_{rel}(x_{rel})$$

where $\psi_{cm}$ is a solution for free particle motion with constant momentum $P_{cm} = p_{total}$ and $\psi_{rel}$ is an eigenfunction for the 1-particle Hamiltonian $\frac{p_{rel}^2}{2\mu} + V(x_{rel})$.