1 Born Approximation

We can use time-dependent perturbation theory for an alternative approach to the scattering problem. Here, we consider only the first order which will yield an approximation. The idea is that we use Fermi's Golden Rule to find the transition rate from a plane wave initial state (along the z-axis) to another plane wave final state within a small range of momenta (Δp) and directions ($\Delta\Omega$) centered on the direction $\hat{r}$ in which our detector is sitting. As we already showed, this will yield a finite answer after integrating over the delta-function. This rate has to be divided by the incoming current density represented by the initial plane wave to arrive at the cross section.

We describe the initial and final states as follows:

$$|i>=|\vec{p_{in}}>; \vec{p_{in}} = p_0\hat{z}$$
$$|f>=|\vec{p_f}>; \vec{p_f} = p_0\hat{r}(\theta,\phi).$$

Applying a perturbation H_p that is not time dependent :

$$H_p = Ve^{i\omega_p t} \quad \omega_p = \omega_{fi} = \frac{E_f^0 - E_i^0}{\hbar} = 0. \tag{1}$$

Using Fermi Golden Rule after integration over $\Delta p, \Delta\Omega$:

$$w_{if} = \frac{2\pi m p_f \delta\Omega}{\hbar}|m_{fi}|^2, \tag{2}$$

with $m_{fi} =< f|H_p|i >$. With proper normalization, the incoming plane wave is

$$< \vec{r}|\vec{p_{in}}>= \frac{1}{(2\pi\hbar)^{3/2}}e^{i\vec{p_{in}}\cdot\vec{r}/\hbar} \tag{3}$$

and the incoming current density is $\vec{J_{in}} = \frac{1}{(2\pi\hbar)^3}\vec{p_{in}}/m$. So the elastic cross section is given by:

$$\frac{d\sigma}{d\Omega} = \frac{w_{if}}{|\vec{J_{in}}|} = (2\pi)^4 m^2\hbar^2|m_{fi}|^2. \tag{4}$$

If $H_p = V(\vec{r})$ we have:

$$\begin{aligned} m_{fi} =< f|V|i > &= \int d^3r \frac{1}{(2\pi\hbar)^{3/2}}e^{-i\vec{p_f}\cdot\vec{r}/\hbar}V(\vec{r})\frac{1}{(2\pi\hbar)^{3/2}}e^{i\vec{p_i}\cdot\vec{r}/\hbar} \\ &= \frac{1}{(2\pi\hbar)^3}\int d^3r e^{-i\vec{q}\vec{r}}V(\vec{r}), \end{aligned} \tag{5}$$

where we have defined the momentum transfer $\vec{q} = \frac{\vec{p_f}-\vec{p_i}}{\hbar}$. In other words, the matrix element is basically the Fourier Transform of the potential with the momentum transfer $\vec{q}$. We note that

$$q^2 = \frac{p_0^2}{\hbar^2}(\hat{r}^2 + \hat{z}^2 - 2\hat{r}\cdot\hat{z}) = \frac{1}{\hbar^2}2p_0^2(1-\cos(\theta)) = \frac{1}{\hbar^2}4p_0^2\sin^2(\theta/2). \tag{6}$$

For now, assume that the potential is rotationally symmetric. Inside the integral, we can choose as the z-direction for the integration variable the direction of $\vec{q}$ such that $\vec{q} = q\hat{z}$ and we have $\vec{q} \cdot \vec{r} = \cos\vartheta$.

$$\begin{aligned} m_{fi} &= \frac{1}{(2\pi\hbar)^3} 2\pi \int_{-1}^{+1} d\cos\vartheta \int_0^\infty r^2 dr e^{-iqr\cos\vartheta} V(r) \\ &= \frac{1}{2\pi^2\hbar^3 q} \int_0^\infty dr r \sin(qr) V(r) \\ &= -\frac{1}{2\pi^2\hbar^3 q} \frac{\partial}{\partial q} \int_0^\infty \cos(qr) V(r) dr. \end{aligned} \tag{7}$$

Using that expression we obtain the following differential cross section:

$$\frac{d\sigma}{d\Omega} = \frac{4m^2}{\hbar^4 q^2} \left(\frac{\partial}{\partial q} \int_0^\infty \cos(qr) V(r) dr \right)^2 \tag{8}$$

Step Potential If we have the following potential:

$$V(x) = \begin{cases} V_0 & r < a \\ 0 & r > a \end{cases} \tag{9}$$

we obtain the following cross section:

$$\begin{aligned} \frac{d\sigma}{d\Omega} &= \frac{4m^2}{\hbar^4 q^2} V_0^2 \left(\frac{\partial}{\partial q} \left(\frac{\sin(qa)}{q} \right) \right)^2 \\ &= \frac{4m^2}{\hbar^4 q^2} V_0^2 \left(\frac{a\cos(qa)}{q} - \frac{\sin(qa)}{q^2} \right)^2 \end{aligned} \tag{10}$$

In the limit $qa \to 0$ the term in brackets can be approximated as

$$\frac{a\cos(qa)}{q} - \frac{\sin(qa)}{q^2} \approx \frac{a}{q} \left(1 - \frac{1}{2}a^2q^2 - (1 - \frac{1}{6}q^2a^2) \right) = -\frac{1}{3}qa^3 \tag{11}$$

and the cross section becomes:

$$\frac{d\sigma}{d\Omega} = \frac{4m^2 V_0^2 a^6}{9\hbar^4}, \tag{12}$$

that gives:

$$f(\theta) = \pm \frac{2mV_0 a^3}{3\hbar^2}. \tag{13}$$

The same result can be gotten from the phase shift analysis in the limit where both the external momentum) k and the internal momentum $k_1 \approx \sqrt{2m|V_0|}$ are small.

Modified Coulomb Potential For the Coulomb potential it is not possible to define the free incoming and outgoing states since the asymptotic behaviour is $V \sim \frac{1}{r}$.
So we consider a modified Coulomb potential $V = \frac{Ze^2 e^{-\mu r}}{r}$ that gives a matrix

element:

$$\begin{aligned}
H_{fi} &= Ze^2 \frac{1}{(2\pi^2\hbar^3)} \int_0^\infty dr \frac{e^{iqr} - e^{-iqr}}{2iq} e^{-\mu r} \\
&= Ze^2 \frac{1}{(2\pi^2\hbar^3) 2iq} \left[\frac{-e^{(iq-\mu)0}}{iq-\mu} - \frac{-e^{(-iq-\mu)0}}{-iq-\mu} \right] \\
&= Ze^2 \frac{1}{(2\pi^2\hbar^3)(q^2+\mu^2)} \qquad (14)
\end{aligned}$$

Hence we have for $\mu \to 0$:

$$\begin{aligned}
\frac{d\sigma}{d\Omega} &= \frac{4Z^2 e^4 m^2}{\hbar^4 q^4} \\
&= \frac{4Z^2 e^4 m^2}{16 p_0^4 \sin^4(\theta/2)} \\
&= \frac{Z^2 \alpha^2 \hbar^2 c^2}{16 E_{in}^2 \sin^4(\theta/2)}. \qquad (15)
\end{aligned}$$