Hydrogen Atom in electromagnetic field of incoming light wave continued

Starting with

$$\widehat{H_p} = \frac{e}{2mc}\vec{A}_o \cdot \vec{P}\, e^{i(\vec{k}\cdot\vec{r})}$$

we now calculate

$$H_{fi} = \frac{e}{2mc}\vec{A}_o \langle \overrightarrow{P_f} | \vec{P} | 100 \rangle = \frac{e}{2mc}\vec{A}_o \overrightarrow{P_f} \langle \overrightarrow{P_f} | 100 \rangle$$

Where $\langle \overrightarrow{P_f} | 100 \rangle$ is the Fourier transform of the ground state wave function which is given by the following,

$$\left(\frac{1}{2\pi\hbar}\right)^{3/2} \sqrt{\frac{1}{\pi a_o}} \int_0^{2\pi} d\varphi \int_{-1}^{1} d\cos\theta \int_0^{\infty} r^2 dr e^{-i\vec{P}_f\cdot\vec{r}/\hbar} e^{r/a_o}$$

Choosing $\vec{P}_f$ in the z-direction

$$\left(\frac{1}{2\pi\hbar}\right)^{3/2} \sqrt{\frac{1}{\pi a_o}} 2\pi \int_{-1}^{1} d\cos\theta \int_0^{\infty} r^2 dr e^{-iP_f r\cos\theta/\hbar} e^{r/a_o}$$

$$= \left(\frac{1}{2\pi\hbar}\right)^{3/2} \sqrt{\frac{1}{\pi a_o}} \frac{2\pi\hbar}{-iP_f} \int_0^{\infty} r dr \left(e^{-iP_f r/\hbar} - e^{iP_f r/\hbar}\right) e^{r/a_o}$$

In the end we get,

$$H_{fi} = \frac{e}{2mc}\frac{1}{\pi^2}\frac{1}{(2\hbar a_o)^{3/2}}\vec{A}_o \cdot \vec{P}_f \frac{8\pi/a_o}{\left(\frac{P_f{}^2}{\hbar^2} + \frac{1}{a_o{}^2}\right)^2}$$

$$H_{fi} = \frac{e}{mc}\frac{1}{\pi}\frac{1}{\left(\frac{\hbar}{a_o}\right)^{3/2}}\vec{A}_o \cdot \vec{P}_f \frac{\sqrt{2}}{\left(\frac{a_o{}^2 P_f{}^2}{\hbar^2} + 1\right)^2}$$

Now we can write the rate as using,

$$R(i \to f) \quad = \frac{2\pi m P_f}{\hbar} \left|H_{fi}\right|^2 \Delta\Omega$$

$$R(i \to f) \quad = \frac{2\pi m P_f}{\hbar}\left(\frac{e}{mc}\frac{1}{\pi}\frac{1}{\left(\frac{\hbar}{a_o}\right)^{3/2}}\vec{A}_o \cdot \vec{P}_f \frac{\sqrt{2}}{\left(\frac{{a_o}^2 {P_f}^2}{\hbar^2}+1\right)^2}\right)^2 \Delta\Omega$$

$$R(i \to f) \quad = \frac{{a_o}^3 P_f \Delta\Omega 4 \mathrm{e}^2 \left(\vec{A}_o \cdot \vec{P}_f\right)^2}{\pi m c^2 \hbar^4}\frac{1}{\left[\left(\frac{a_o P_f}{\hbar}\right)^2+1\right]^4}$$

Let's assume a detector at an angle of θ from the source,

What is the probability that an electron get kicked out and the detector detect it

$\Delta\Omega = \sin\theta\, \Delta\theta\Delta\varphi = \Delta\cos\theta\, \Delta\varphi$. If we cover the whole azimuth ($\Delta\varphi = 2\pi$)

$$Rate(\cos\theta \,..\cos\theta + \Delta\cos\theta) = \frac{8{a_o}^3 {P_f}^3 \mathrm{e}^2 \mathrm{A}_0^{\ 2}}{mc^2\hbar^4}\frac{\cos^2\theta}{\left[\left(\frac{a_o P_f}{\hbar}\right)^2+1\right]^4}\Delta\cos\theta$$

So we get maximum probability at $\theta = 0 \; and \; 180$

What is the total probability that an electron gets kicked out per unit time ?

$$Rate\big(|100\rangle \to |P_f\rangle\big) = \frac{16{a_o}^3 {P_f}^3 \mathrm{e}^2 \mathrm{A}_0^{\ 2}}{3mc^2\hbar^4}\frac{1}{\left[\left(\frac{a_o P_f}{\hbar}\right)^2+1\right]^4}$$

Is this a reasonable rate ? See homework.

The pointing vector for this problem,

$$\vec{S} = \frac{c}{4\pi}\left(\vec{E}\times\vec{B}\right)$$

Using what we found earlier,

$$\vec{S} = \frac{c}{4\pi}\left(\frac{\omega}{c}\right)^2 {A_o}^2 \; \cos^2\!\left(\vec{k}\cdot\vec{r} - \omega t\right)$$

To average,

$$\langle \vec{S} \rangle = \frac{\omega^2}{8\pi c} {A_0}^2$$

So we can write the energy per unit time equals to pointing vector in some area

$$\frac{E_{out}}{\Delta t} = R\big(|100\rangle \to |P_f\rangle\big)\hbar\omega_\mathrm{p} = S \cdot \Delta\sigma$$

$$\Delta\sigma = \frac{E_{out}/_t}{S}$$

Using what we found earlier we can finally write this as,

$$\Delta\sigma = \frac{E_{out}/_t}{S} = \frac{128\pi \mathrm{e}^2\ {a_o}^3 {P_f}^3}{3mc \quad \omega_\mathrm{p}\ \hbar^3} \frac{1}{\left[\left(\frac{a_o P_f}{\hbar}\right)^2 + 1\right]^4}$$