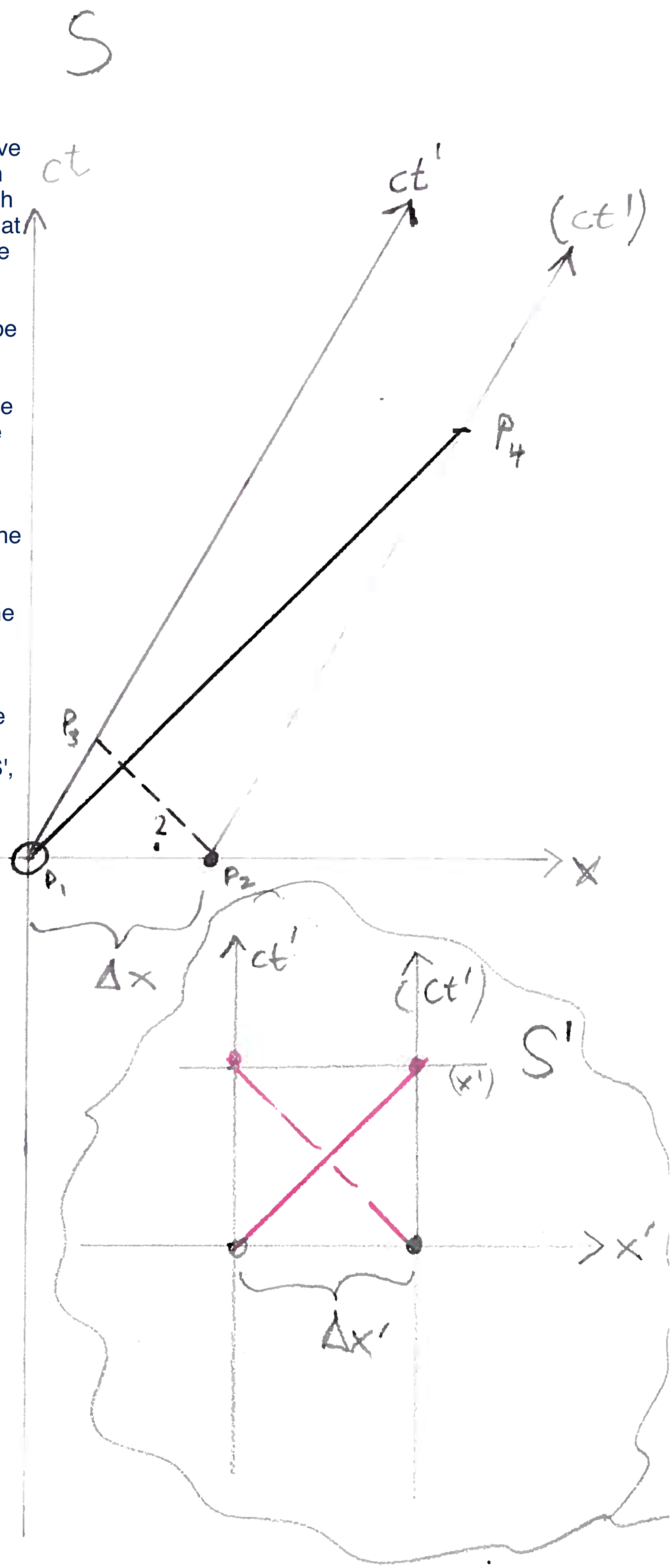
Light flashes go off at P1 and P2, simultaneously as measured by S, namely at ct = 0. One flash is at x = 0, the other at a distance Delta-x to the right.

System S’ is moving with 57.7% of the speed of light to the right. The (darker) ct’ axis indicates where its origin is (relative to S) for each point in time t. (The angle the axis makes with the ct axis = the vertical is given by tan(alpha) = 0.577, which yields alpha = 30 degrees.) Both systems are synchronized at the origin, meaning that the “event” x’=0, ct’ = 0 occurs at the point x=0, ct=0 as measured in S.

QUESTION: Will the two light flashes also be measured to be simultaneous in S’?

ANSWER: NO, that is not possible IF the speed of light is the same, c, in S’ as in S. (The paths of the two light flashes are indicated byt the heavy line and the dashed line at 45 degrees).

PROOF: Assume S’ has a second clock, some distance to the right of the first one, both synchronized to each other in S’. That 2nd clock is indicated by the (faint) parallel axis also labeled as ct’, and it just happens to pass the point where the 2nd flash goes off. IF these two points were simultaneous in S', then the arrival times of the two light flashes at the OPPOSITE ct' axes would have to be ALSO simultaneous (see inset, which shows the whole situation how it would like from the point of view of S' if the two light flashes were simultaneous). However, if P1 and P2 are simultaneous in S', then P3 and P4 could not possibly be also simultaneous - they cannot be connected by a line parallel to the one connecting P1 and P2.



From this deliberation, it is “obvious” (hopefully) that any point to the right of the origin in S’ that is simultaneous with the origin has to be at a later time ct as measured in S. (If you move P2 up along the faint ct’ axis, you can see how the point P3 also becomes later, until the line connecting P3 with P4 will be parallel to that connecting P1 and P2 - see next page).

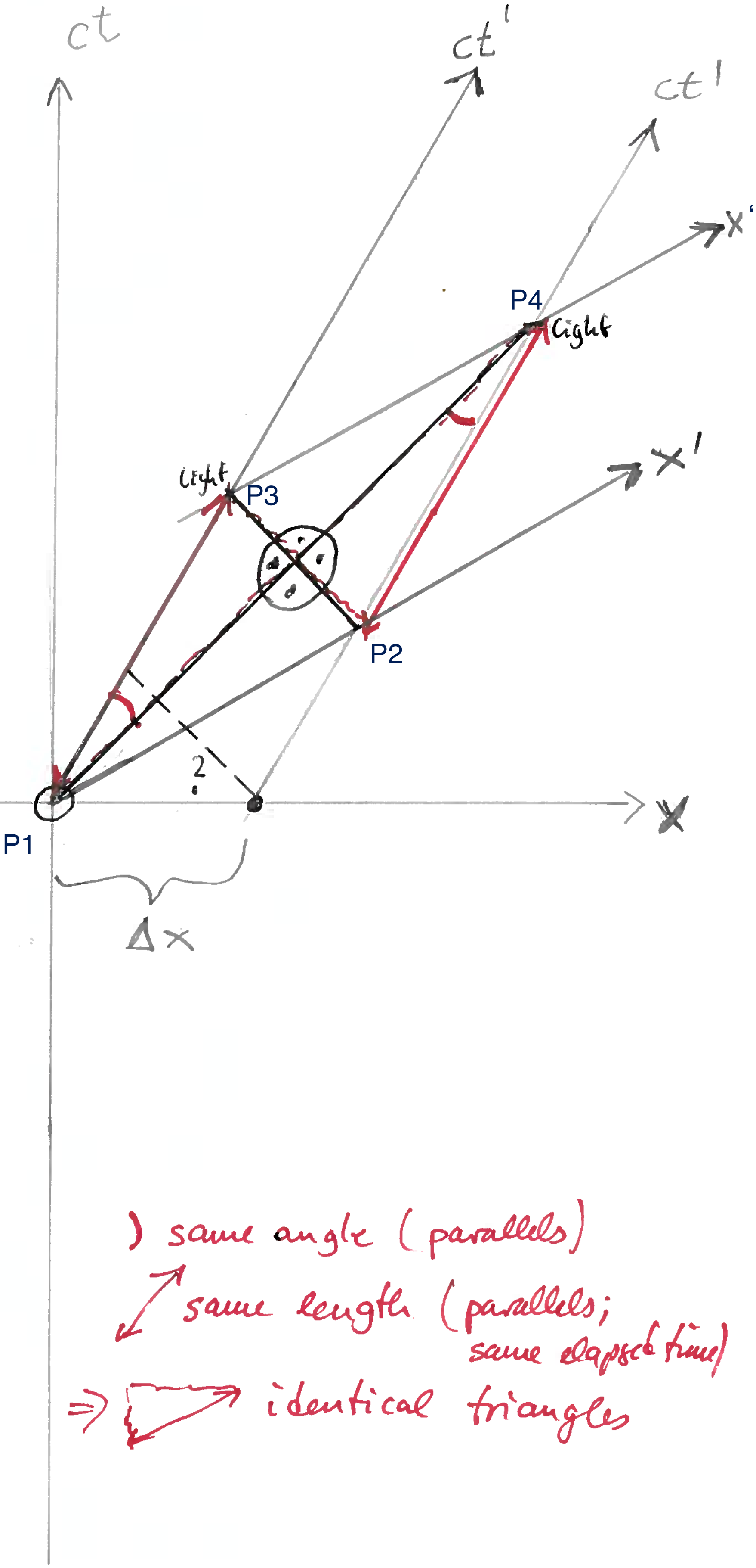
The practical interpretation is clear: Since S’ is moving towards the right, a light flash going off at P2 would appear to be EARLIER than that at P1, since S’ is moving TOWARDS that light flash from P2 and AWAY from that in P1. Therefore, two events that are simultaneous in S CAN NOT be simultaneous in S’.

This is the correct version of two events P1 and P2 being simultaneous in S' (corresponding to the picture in the inset on the previous page). The lower x' axis connects ALL events that are simultaneous with the origin in S', and therefore the upper x' axis, which is parallel to it, ALSO connects events that are simultaneous in S'. You can see that both the emission of the two light pulses (P1 and P2) and the observation at the opposite clocks (P3 and P4) are now simultaneous in S'. Of course, P2 is much "later" in S than P1 - so S and S' do not agree on what is simultaneous.

QUESTION: What angle does the x' axis make with the x axis?

ANSWER: The same angle (arctan v/c) as ct' makes with the ct axis!

PROOF: Begin by considering the two triangles outlined in red. They have both the same angles (two right angles as indicated and two vertex angles from a single line crossing two parallels). The two red sides must also be equal (parallels crossed by parallels). Therefore they are equal.



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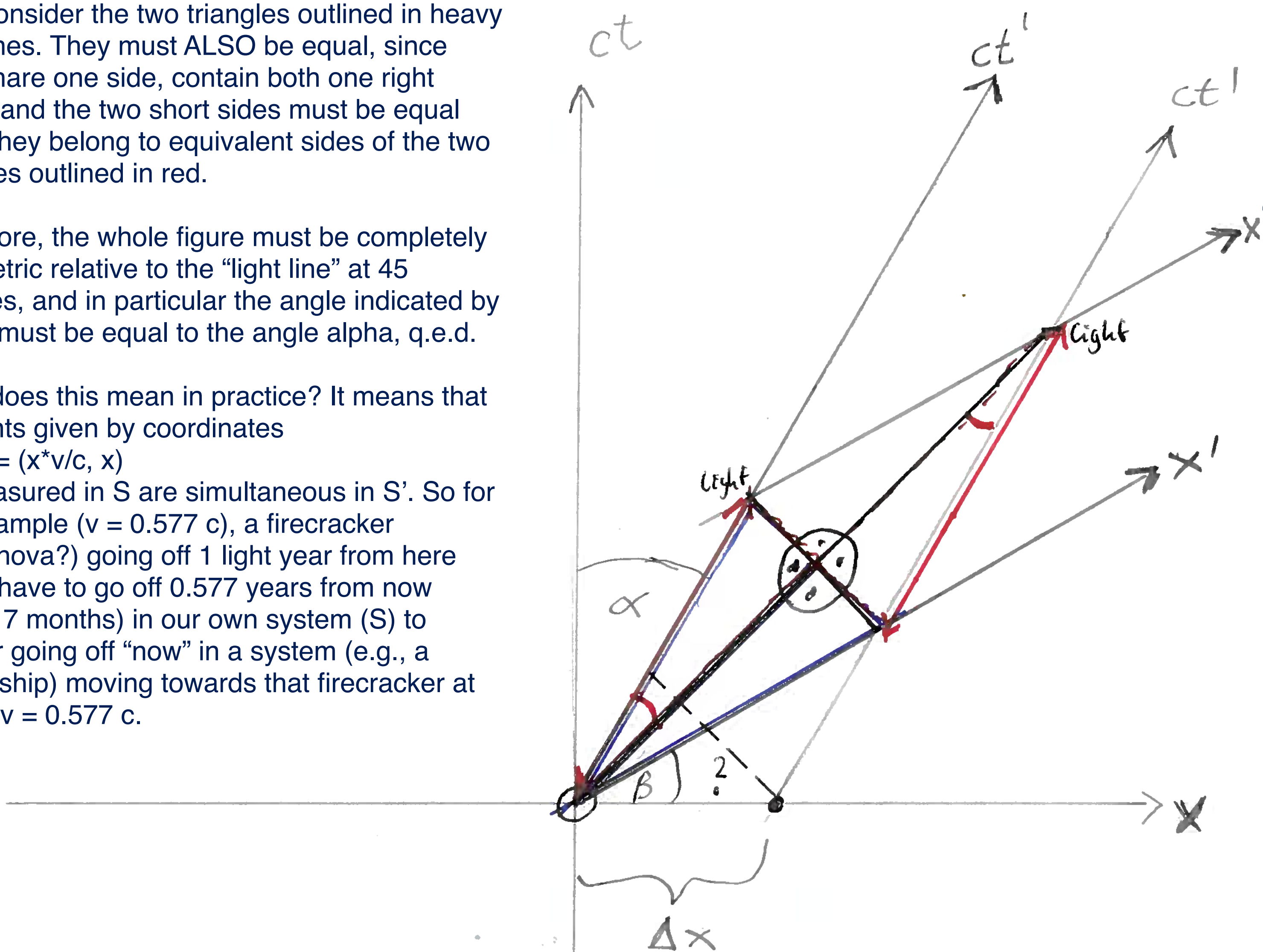
Now consider the two triangles outlined in heavy dark lines. They must ALSO be equal, since they share one side, contain both one right angle, and the two short sides must be equal since they belong to equivalent sides of the two triangles outlined in red.

Therefore, the whole figure must be completely symmetric relative to the "light line" at 45 degrees, and in particular the angle indicated by "beta" must be equal to the angle alpha, q.e.d.

What does this mean in practice? It means that all points given by coordinates
(ct, x) = (x*v/c, x)
as measured in S are simultaneous in S'. So for our example (v = 0.577 c), a firecracker (supernova?) going off 1 light year from here would have to go off 0.577 years from now (about 7 months) in our own system (S) to appear going off "now" in a system (e.g., a space ship) moving towards that firecracker at speed v = 0.577 c.



same angle (parallels)

same length (parallels; same elapsed time)

⇒ identical triangles

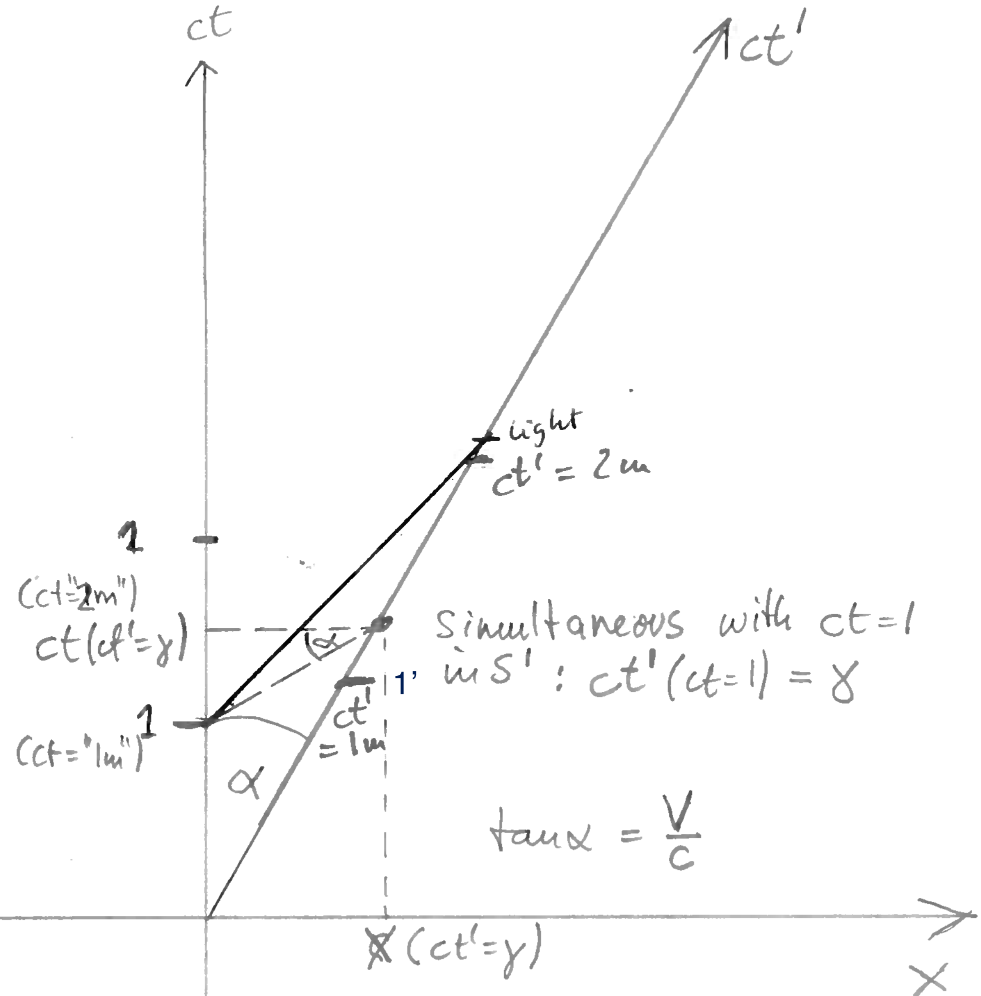
identical triangles

$\Rightarrow \alpha = \beta = \arctan \frac{v}{c}$

2.) Time Scale

The time scale in S' must also be different from that in S, given that both systems are totally equivalent in their description of nature (none is "preferred" or "absolutely at rest").

Assume that the clock in S ticks of "ct=1m" at the horizontal mark "1". Assume that the clock in S' ticks of "ct'=1m" at the horizontal mark 1', which corresponds to some later time ct=gamma. (Our task is to find the value of gamma). Vice versa, the time of "ct=1m" as measured by S' is even later than that, because horizontal mark 1 is simultaneous with the black blob on the ct' axis, according to S'. So it occurs at ct'=gamma according to S, but because all times in S are delayed by a factor gamma according to S, this blob really only happens at the time ct = gamma^2. The rest is just geometry and algebra.



$$x(ct'=\gamma) = \tan\alpha \cdot ct(ct'=\gamma)$$

$$ct(ct'=\gamma) - 1 = \tan\alpha \cdot x(ct'=\gamma) = \tan^2\alpha \cdot ct(ct'=\gamma)$$

$$\Rightarrow ct(ct'=\gamma) = \frac{1}{1-\tan^2\alpha} = \frac{1}{1-V^2/c^2}$$

S and S' symmetric:

if in S', γ "m" elapse before 1 m elapses in S,

then in S, γ^2 "m" elapse before γ m elapse in S'

$$\Rightarrow \gamma^2 = \frac{1}{1-V^2/c^2}, \quad \gamma = \frac{1}{\sqrt{1-V^2/c^2}}$$

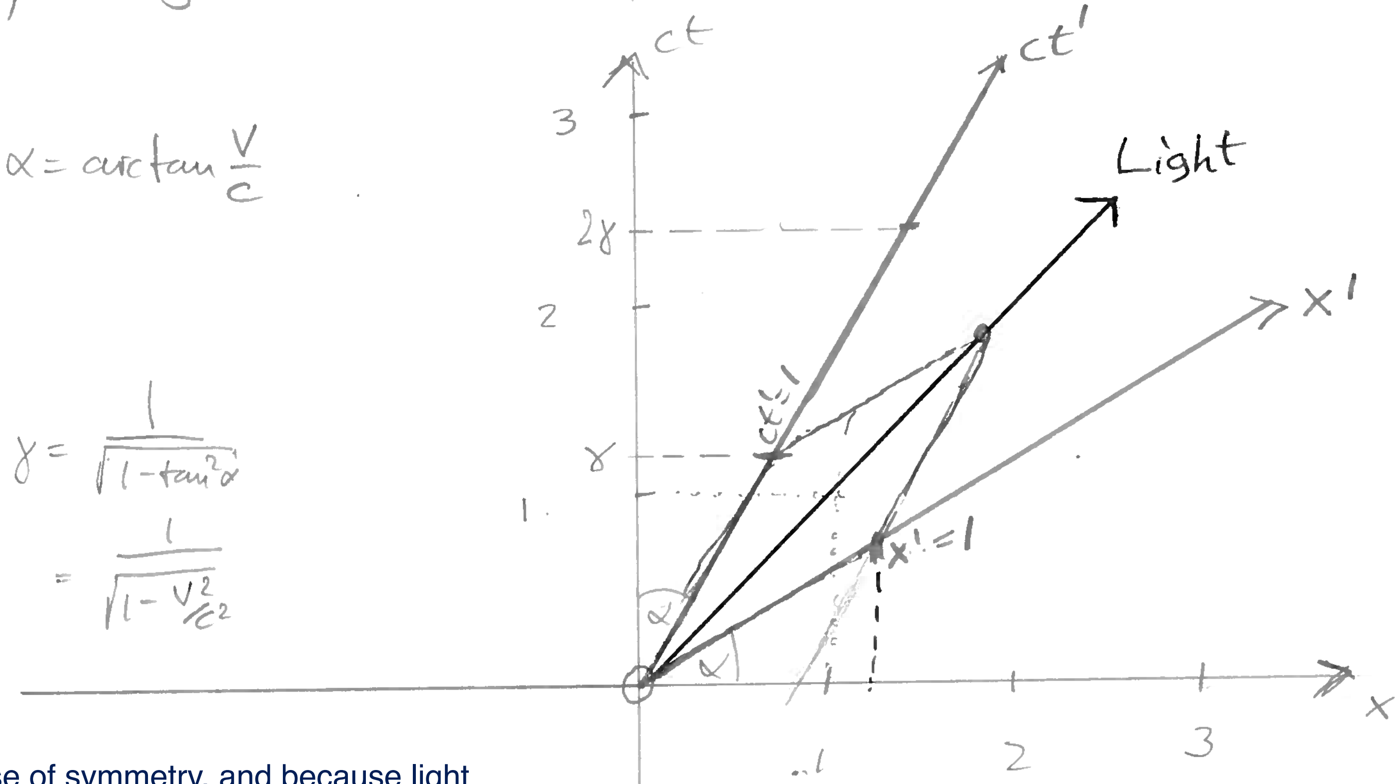
$\Rightarrow$ according to S, the clock in S' goes slow by factor γ \
" " S', the clock in S " " " γ } No contradiction.

3) Length Scale

$\alpha = \arctan \frac{V}{c}$

$\gamma = \frac{1}{\sqrt{1-\tan^2\alpha}}$

$= \frac{1}{\sqrt{1-V^2/c^2}}$



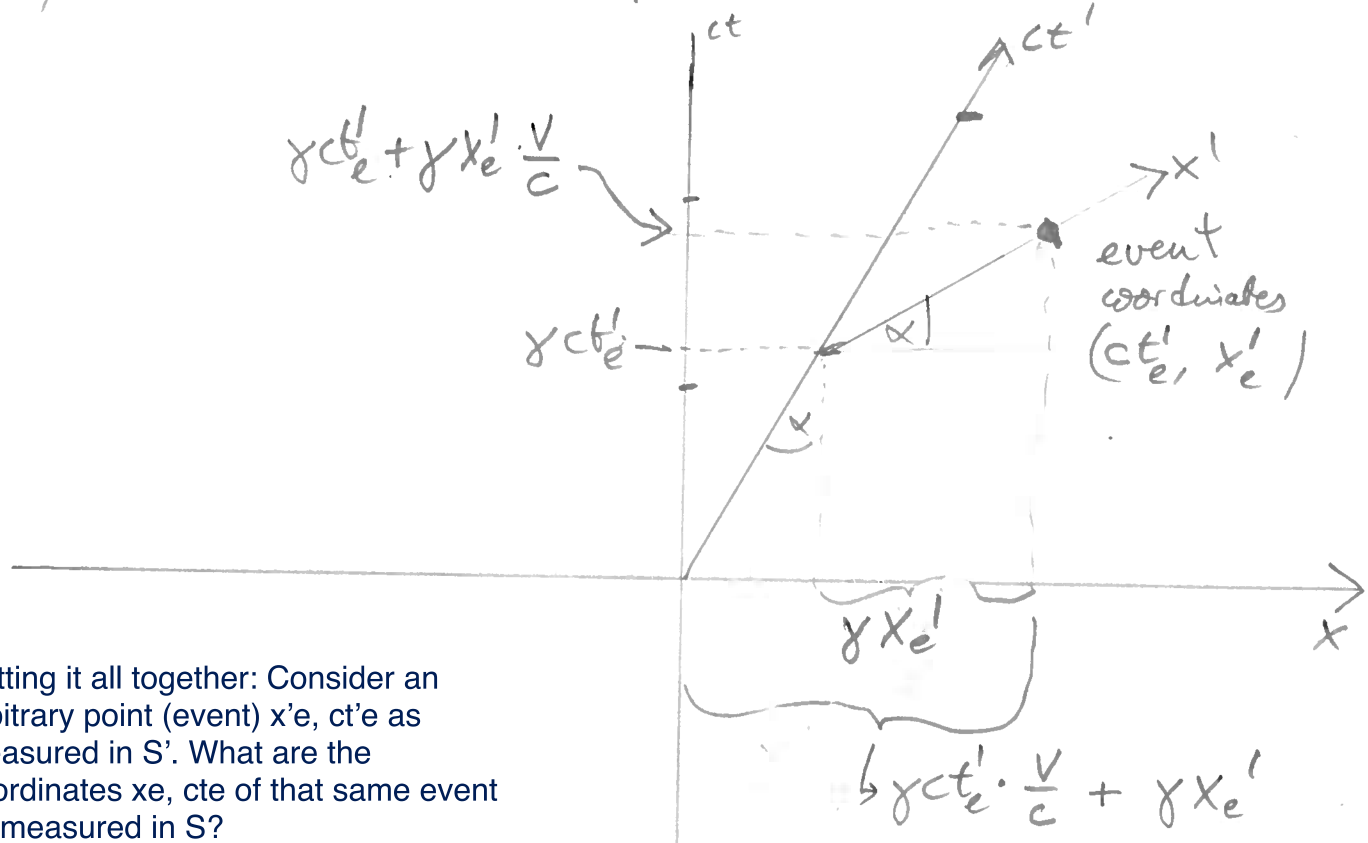
Because of symmetry, and because light must have the same speed c in S' as in S, the point x' = 1 must be at x = gamma just like ct' = 1 is at ct = gamma. The dot on the light trajectory shows at what point S' measures the light to have traveled by 1 m from the origin in S'.

For the same reason (S and S' are completely equivalent), the position "x=1" in S appears to be crossing the x' axis at a distance 1/gamma from the origin. So 1 m in S appears shortened to 1/gamma m in S' -> "moving sticks are shortened".

$\Rightarrow x(x'=1) = \gamma$ also!

Note: from S', a meter stick in S of length γ appears only to have length 1 $\Rightarrow$ "moving sticks are shortened"

4.) Full Lorentz transformation



Putting it all together: Consider an arbitrary point (event) x’e, ct’e as measured in S’. What are the coordinates xe, cte of that same event as measured in S?

Straightforward geometry and algebra -> Lorentz transformations

$$x_e = \gamma\left(ct_e' \cdot \frac{V}{c} + x_e'\right) = \frac{x_e' + \frac{V}{c} c t_e'}{\sqrt{1 - V^2/c^2}}$$

$$ct_e = \gamma\left(ct_e' + \frac{V}{c} x_e'\right) = \frac{ct_e' + \frac{V}{c} x_e'}{\sqrt{1 - V^2/c^2}}$$

$$\Rightarrow \quad t_e = \frac{t_e' + \frac{V}{c^2} x_e'}{\sqrt{1 - V^2/c^2}}$$

What about x', ct' in terms of x, ct?

can invert algebraically (HW)

or symmetry: $\frac{V}{c} \rightarrow -\frac{V}{c}$

$$\Rightarrow \quad x_e' = \frac{x_e - V t_e}{\sqrt{1 - V^2/c^2}}, \quad t_e' = \frac{t_e - \frac{V}{c^2} x_e}{\sqrt{1 - V^2/c^2}}$$