Lecture 15-16

1) Summary QM of particle in 1D

→ state vector $|\psi\rangle$ represented by $\psi(x)$ $(-\infty < x < \infty)$
at every time t $|\psi\rangle(t) \longleftrightarrow \psi(x,t)$

Normalized: $\langle\psi|\psi\rangle = \int_{-\infty}^{\infty} \psi^*(x)\,\psi(x)\,dx = 1$

a) Evolution in time: $i\hbar \frac{\partial}{\partial t}|\psi\rangle(t) = H\,|\psi\rangle(t)$ $\quad H = \frac{p^2}{2m} + V$

$$i\hbar \frac{\partial}{\partial t}\psi(x,t) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x,t) + V(x)\,\psi(x,t)$$

"stationary solution": $|\psi\rangle_E(0) =$ EV to H with $H|\psi_E\rangle(0) = E|\psi_E\rangle(0)$

⇒ true for all times t!

$t=0$: $-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi_E(x) + V(x)\psi_E(x) = E\,\psi(x,t)$

⇒ time evolution: $|\psi\rangle(t) = |\psi_E\rangle(0)\cdot e^{-i\frac{E}{\hbar}t}$

General solution: $\psi(x,t) = \sum_E a_E\,\psi_E(x)\,e^{-i\frac{E}{\hbar}t}$

b) Probability to find particle at $x \dots x+\Delta x$ (Δx 'small')

$$\Delta P(x \dots x+\Delta x) = |\psi(x,t)|^2\,\Delta x$$

In general:

Operator Ω → EV's ω_i; can only find ω_i when measure; collapse → $|\psi\rangle_{\omega_i}$

Probability $|\langle\psi_{\omega_i}|\psi\rangle|^2$

Ex.: ∞ square well
$\psi_n(x) = \sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L}$

$E_n = \frac{n^2\pi^2\hbar^2}{2mL^2}$

$P(x \dots x+\Delta x) = \frac{2}{L}\sin^2\frac{n\pi x}{L}\Delta x$

refer to mid term.

Note: $\int_0^L |\psi_n(x)|^2 dx = 1$!

Expectation value: $\langle\Omega\rangle = \langle\psi|\Omega|\psi\rangle$

2) 3 D : generalize $|\psi\rangle \xrightarrow{(t)} \psi(\vec{r},t)$ $\vec{r} \in \mathbb{R}^3$

Normaliz. $\langle\psi|\psi\rangle = \iiint d^3\vec{r}\, |\psi(\vec{r})|^2 = 1$

a) Hamiltonian: $\frac{P_x^2 + P_y^2 + P_z^2}{2m} + V(\vec{r}) \rightarrow -\frac{\hbar^2}{2m}\vec{\nabla}^2 + V(\vec{r})$

still can find separable (stationary) solutions $\cdot e^{-i\frac{E}{\hbar}t}$ $\mathcal{S}_{ES}$ w/ EV E

b) $\Delta\mathrm{Prob}(\vec{r}, \vec{r} + \Delta\tau$ ← small volume surrounding $\vec{r}$ $) = |\psi(\vec{r})|^2 \Delta\tau$

Example: 3D square box L^3: combine solutions

$\psi_E(\vec{r}) = \varphi_k(x)\cdot\varphi_m(y)\cdot\varphi_n(z) = \sqrt{\frac{8}{L^3}} \sin\frac{k\pi x}{L} \sin\frac{m\pi y}{L} \sin\frac{n\pi z}{L}$ ← point out separable.

energy $E_{k,m,n} = \frac{\pi^2\hbar^2}{2mL^2}(k^2 + m^2 + n^2)$ go through detail!

Note: new phenomenon → degenerate EVs!

eg. $k = m = n = 1$ ground state (only one)

but $k = 2, m = n = 1$ or $k = 1, m = 2, n = 1$ or $k = m = 1, n = 2$ all give same energy! (but different ES!)

c) Example: g.s. → Prob for center? all sin = 1, ⇒ $\frac{8\tau}{L^3}$

edge (surface)? 0!

3) What if $V(\vec{r}) = V(r)$ → only depends on distance from origin? → spherical coordinates r, θ, φ → EXPLAIN

$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \rightarrow \frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{1}{r^2}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right)$

multiply with $-\frac{\hbar^2}{2m}$ for H! (*)

3 cont'd: simple "exact" problem: set $r=$ const., $\sin\theta = 1$

$\Rightarrow |\psi\rangle = \psi(\varphi) \rightarrow$ solve $-\frac{\partial^2}{\partial\varphi^2}\psi(\varphi) = m^2\varphi(\varphi)$

answ.: $\psi(\varphi) = e^{\pm im\varphi}$ but $\varphi(2\pi+\varphi) = \psi(\varphi) \Rightarrow m =$ int

$(0, \pm 1, \pm 2, \ldots)$. Turns out $\frac{\hbar}{i}\frac{\partial}{\partial\varphi}$ represents L_z

and therefore $\psi_m(\varphi)$ are ES to L_z w/ EV $m\hbar$

$\rightarrow$ angular momentum is quantized! (Another one...)

Turns out the whole expression $-\hbar^2 \cdot (*) = \vec{L}^2$!

(not obvious) Solutions are simultaneous EV to both

$|\vec{L}|^2$ (total angular mom.) and L_z, w/ EV

$\hbar^2 \ell(\ell+1)$ and $\hbar m$, respectively. They are called

spherical harmonics and are written $Y_{\ell m}(\theta, \varphi)$

ES: $\vec{L}^2 Y_{\ell m} = \hbar^2 \ell(\ell+1) Y_{\ell m}$ $\quad \ell = 0, 1, 2 \ldots$

(called "angular mom. l")

$L_z Y_{\ell m} = \hbar m \, Y_{\ell m}$

$m = 0, \pm 1, \ldots \pm \ell$ (max!)

since m cannot be $> \ell$

Properly normalized: $Y_{00}(\theta,\varphi) = \frac{1}{\sqrt{4\pi}}$ = const $= \frac{1}{\sqrt{4\pi}} P_0(\cos\theta)$

(clearly, $\vec{L}^2 Y_{00} = L_z Y_{00} = 0$! $\rightarrow \ell = m = 0$)

$Y_{10} = \sqrt{\frac{3}{4\pi}} \cos\theta = \sqrt{\frac{3}{4\pi}} P_1(\cos\theta)$ $\quad Y_{11} = -\sqrt{\frac{3}{8\pi}} \sin\theta \, e^{i m\varphi}$ ($m=1$)

$Y_{1-1} = \sqrt{\frac{3}{8\pi}} \sin\theta \, e^{-im\varphi}$ ($m=1$) $\quad Y_{20} = \sqrt{\frac{5}{16\pi}} (3\cos^2\theta - 1) = \sqrt{\frac{5}{4\pi}} P_2(\cos\theta)$ etc.