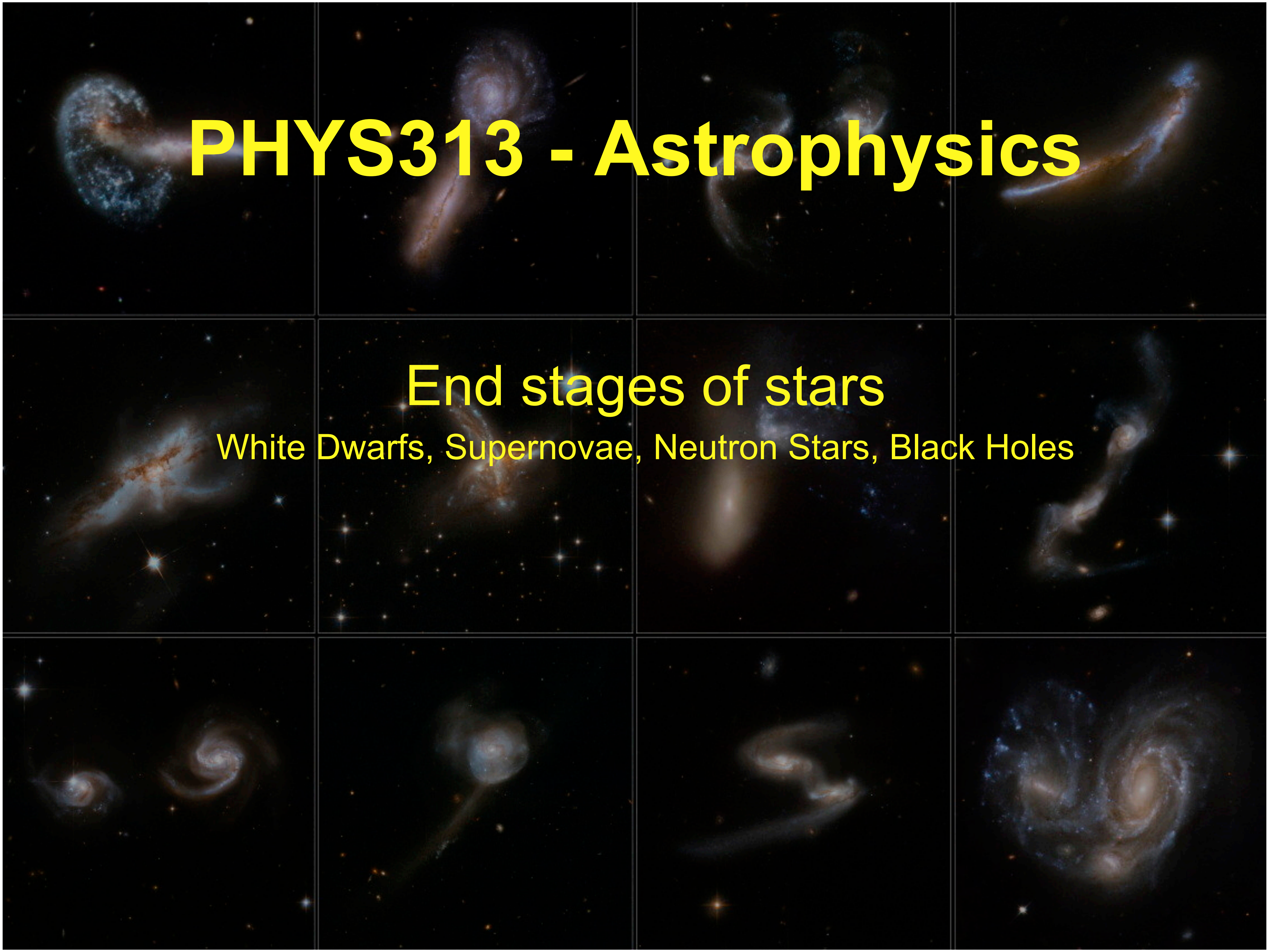
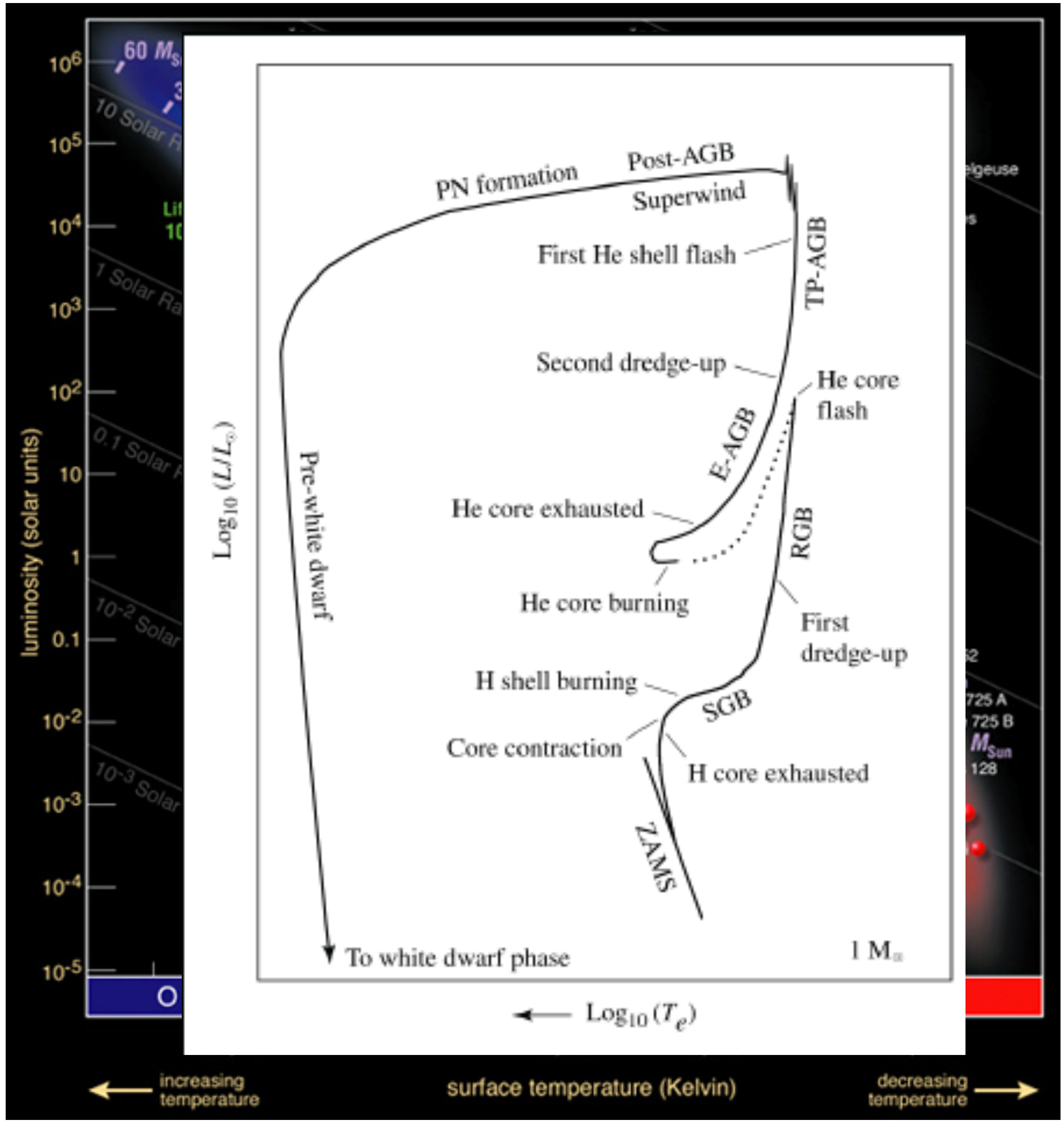

PHYS313 - Astrophysics
End stages of stars
White Dwarfs, Supernovae, Neutron Stars, Black Holes

Additional Resources

- Dan Maoz: “Astrophyics in a Nutshell”
- http://ww2.odu.edu/~skuhn/PHYS313/Home313.html
- http://nicadd.niu.edu/~bterzic/PHYS652/index.htm

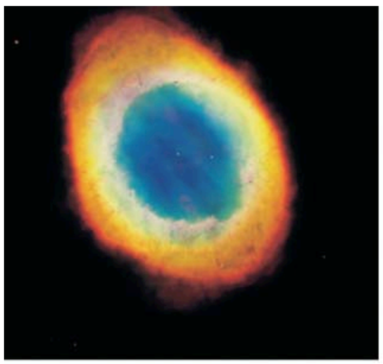
Giant Stars

- Reminder: Last stage of stars after completing Main Sequence existence
  - Core collapses
  - Outer envelope increases enormously
- Sun-like stars: Subgiants => Giants
- Much more massive stars: Supergiants

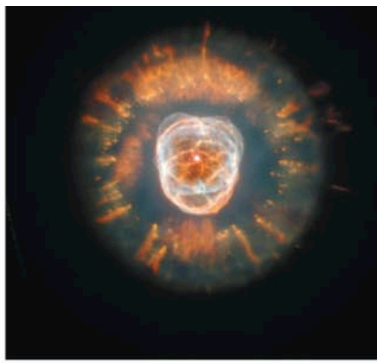


Final Stages of Giants ($\approx M_{\odot}$)

- Final C core collapse
- Shock wave
- Outer layers ejected
- “Planetary” Nebulae



Ring Nebula

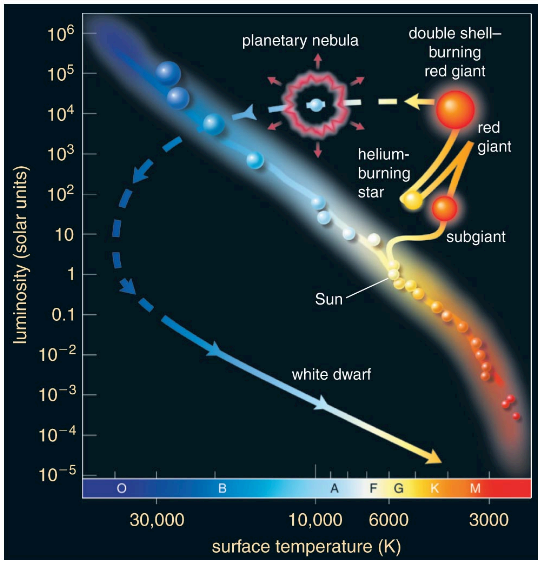


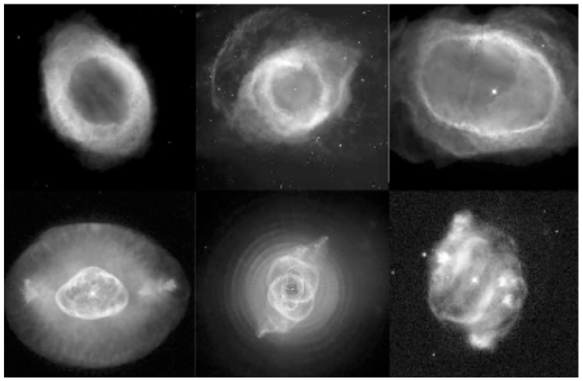
b Eskimo Nebula

Spirograph Nebula

d Hourglass Nebula

White Dwarfs





- Reminder: Last stages of sun and similar-sized stars

  Last stage: Helium burning stops, core collapses and significant fraction of mass gets ejected as planetary nebula

- What happens with the core after the final collapse? => White Dwarf! (Example: Sirius B)
  - Core contracts until "Fermi pressure" of electrons balances gravitational attraction
  - Final size typically <1% of present solar radius => Density 10^6 times larger than that of the sun! Temperature 10^7 K at center

Hydrostatic Equilibrium

We now present a simple model for a star in hydrostatic equilibrium.

Consider a think shell within a star in equilibrium. There are inward force acting on the shell due to its gravitating mass and the outward force of gas pressure:

$$\begin{aligned} F_g &= -G\frac{M(r)\left[\rho(r)4\pi r^2 dr\right]}{r^2} \\ F_p &= 4\pi r^2\left[P(r+dr)-P(r)\right] = 4\pi r^2 dP \end{aligned} \tag{445}$$

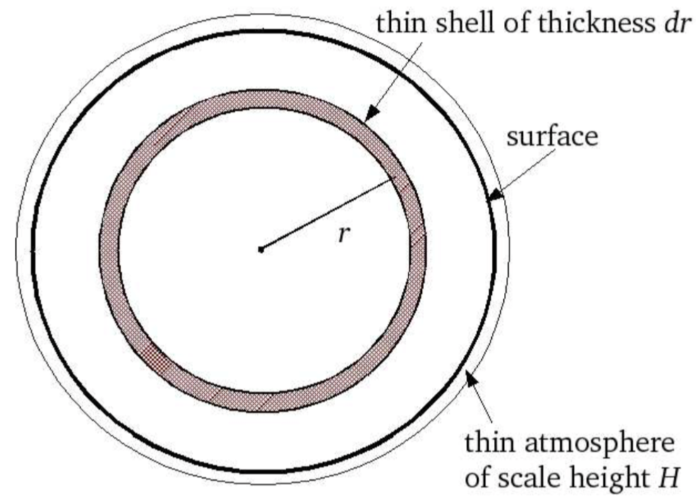
where $M(r)$ is mass interior to the shell:

$$M(r) = 4\pi\int_0^r \rho(\tilde{r})\tilde{r}^2 d\tilde{r}. \tag{446}$$

In hydrostatic equilibrium, these two forces are balanced, so

$$\begin{aligned} F_p &= F_g \\ 4\pi r^2 dP &= -G\frac{M(r)\left[\rho(r)4\pi r^2 dr\right]}{r^2} \\ \Longrightarrow \quad \frac{dP}{dr} &= -\rho(r)\frac{GM(r)}{r^2}. \end{aligned}$$

The equation above is the equation of *hydrostatic equilibrium.*



Example: Sirius B

- Visual companion of Sirius A, 50 yr orbit
  - $M = M_{sun}$
  - T = 27,000 K, Lumi = 3% of sun => $R = 0.008\ R_{sun}$ = 5500 km
  - => density = $2 \cdot 10^6$ x density(sun) = $3 \cdot 10^9$ kg/m^3; 10^{57} nucleons $2 \cdot 10^{36}$ nucleons/m^3 , 10^{36} e$^-$/m^3; Atoms <1/20 of radius apart
- Pressure at center:

$$dP = -\frac{GM}{r^2}\rho\, dr \approx -\frac{G}{r^2}\frac{4\pi r^3}{3}\rho^2 dr = -\frac{4\pi G\rho^2}{3} r\, dr \Rightarrow$$

$$P(R) - P(0) = -\frac{4\pi G\rho^2}{3}\int_0^R r\, dr = -\frac{4\pi G\rho^2}{3}\frac{R^2}{2} \Rightarrow \quad P(0) \approx \frac{2\pi G\rho^2 R^2}{3} \approx 3.9 \cdot 10^{22}\ \mathrm{N/m^2}$$

  - Ideal Gas: $P = nRT/V \approx 1.4 \cdot 10^{13}$ N/m$^2 \cdot T$/K => several orders of magnitude missing. Solution? => Degenerate Fermi-Gas

4.2.1 *Matter at Quantum Densities*

We saw in the previous section that when the core of a star exhausts its nuclear energy supply, it contracts and heats up until reaching the ignition temperature of the next available nuclear reaction, and so on. After each contraction, the density of the core increases. At some point, the distances between atoms will be smaller than their de Broglie wavelengths. At that point, our previous assumption of a classical (rather than quantum) ideal gas, which we used to derive the equation of state, becomes invalid. To get an idea of the conditions under which this happens, recall that the de Broglie wavelength of a particle of momentum p is

$$\lambda = \frac{h}{p} = \frac{h}{(2mE)^{1/2}} \approx \frac{h}{(3mkT)^{1/2}}, \tag{4.9}$$

$$\rho_q \approx \frac{m_p}{(\lambda/2)^3} = \frac{8m_p(3m_e kT)^{3/2}}{h^3}. \tag{4.10}$$

For example, for the conditions at the center of the Sun, $T = 15 \times 10^6$ K, we obtain

$$\rho_q \approx \frac{8 \times 1.7 \times 10^{-24}\ \text{g}\ (3 \times 9 \times 10^{-28}\ \text{g} \times 1.4 \times 10^{-16}\ \text{erg K}^{-1} \times 15 \times 10^6 \text{K})^{3/2}}{(6.6 \times 10^{-27}\ \text{erg s})^3}$$

$$= 640\ \text{g cm}^{-3} \tag{4.11}$$

Fermi-Dirac Distribution

The **Fermi-Dirac phase–space distribution**, embodying these principles for an ideal gas of fermions, is

$$dN = \frac{2s+1}{\exp\left(\frac{E-\mu(T)}{kT}\right)+1}\frac{d^3\mathbf{p}dV}{h^3}, \tag{4.16}$$

$$dN(p)dp = \begin{cases} 2\times 4\pi p^2 \frac{dpdV}{h^3} & \text{if } |\mathbf{p}| \le p_f \\ 0 & \text{if } |\mathbf{p}| > p_f \end{cases}, \tag{4.17}$$

where p_f, called the **Fermi momentum**, is the magnitude of the momentum corresponding to the **Fermi energy** E_f. Dividing by dV, we obtain the number density of electrons of a given momentum p:

$$n_e(p)dp = \begin{cases} 8\pi p^2 \frac{dp}{h^3} & \text{if } |\mathbf{p}| \le p_f \\ 0 & \text{if } |\mathbf{p}| > p_f \end{cases}. \tag{4.18}$$

Integrating over all momenta from 0 to p_f gives a relation between the electron density and p_f:

$$n_e = \int_0^{p_f} \frac{8\pi}{h^3} p^2 dp = \frac{8\pi}{3h^3} p_f^3. \tag{4.19}$$

Non-Relativistic Degeneracy

$$P = \frac{1}{3}\int_0^\infty n(p) p v \, dp. \tag{4.23}$$

Replacing the Maxwell-Boltzmann distribution for $n(p)$ recovers the classical equation of state,

$$P = nkT. \tag{4.24}$$

For a nonrelativistic[3] **degenerate** electron gas, however, we replace $n(p)$ with the Fermi-Dirac distribution in the degenerate limit (Eq. 4.18). Taking $v = p/m_e$, we obtain instead

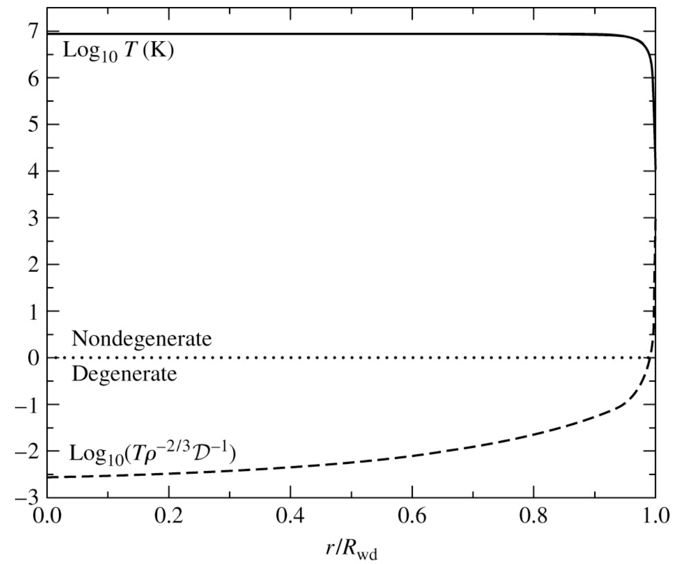
$$P_e = \frac{1}{3}\int_0^{p_f} \frac{8\pi}{h^3}\frac{p^4}{m_e} dp = \frac{8\pi}{3h^3 m_e}\frac{p_f^5}{5} = \left(\frac{3}{8\pi}\right)^{2/3}\frac{h^2}{5m_e} n_e^{5/3}, \tag{4.25}$$

Substituting into Eq. 4.25, we obtain a useful form for the **equation of state of a degenerate nonrelativistic electron gas**:

$$P_e = \left(\frac{3}{\pi}\right)^{2/3}\frac{h^2}{20 m_e m_p^{5/3}}\left(\frac{Z}{A}\right)^{5/3}\rho^{5/3}. \tag{4.27}$$

White Dwarf Structure

- Center (most of volume):
  - High density, degenerate Fermi gas
  - Uniform temperature (high heat conductance)
    - initially 10^9 K (from collapse), quickly cools to a few 10^6 - 10^7 K
  - mostly C, O
- Shell (thin layer, 1% in R):
  - hydrogen, helium
  - insulates star, much lower T -> much reduced radiation ($\propto T_{core}^{7/2}$)
  - further slowdown due to crystallization
  - Oldest white dwarfs have cooled to about 3500K -> can estimate age of galaxy to 10^{10} yr



Interlude: Fermi Gas

- <u>Pauli exclusion principle:</u> No two fermions (spin 1/2 particles) can be in the same quantum state
- <u>Heisenberg uncertainty principle:</u> $\Delta p \cdot \Delta x \approx \hbar$ => two states are indistinguishable if they occupy the same "cell" $dV \cdot d^3p = h^3$ in "phase space" (except for factor 2 because of spin degree of freedom) => for volume V and "momentum volume" $d^3p = 4\pi p^2 dp$ we find for the Number of states between $p \ldots p+dp$:

$$dN = 2\frac{V}{h^3}4\pi p^2 dp = \frac{V}{\pi^2\hbar^3}p^2 dp \quad \Rightarrow \quad N_{tot} = \frac{V}{\pi^2\hbar^3}\frac{p_f^3}{3} \quad \Rightarrow \quad p_f = \hbar\left(3\pi^2\right)^{1/3} n^{1/3}; \quad n = \frac{N_{tot}}{V}; \quad N_{tot} = \frac{M_{star}}{1\text{ g}}\frac{N_A}{2}$$

  - Sirius B: p_f = 670 keV/c for electrons (semi-relativistic - m_e = 511 keV/c^2!)
- – total kinetic energy:

$$E_{tot}^{kin} = \int_0^{p_f} E(p)\frac{V}{\pi^2\hbar^3}p^2\,dp = \begin{cases} \displaystyle\int_0^{p_f}\frac{p^2}{2m}\frac{V}{\pi^2\hbar^3}p^2\,dp = \frac{1}{2m}\frac{V}{\pi^2\hbar^3}\frac{p_f^5}{5} = \frac{3}{5}N_{tot}\frac{p_f^2}{2m} = \frac{3\hbar^2}{10m}N_{tot}\left(3\pi^2\right)^{2/3}\left(\frac{N_{tot}}{V}\right)^{2/3} = \frac{3\hbar^2\left(\frac{9\pi}{4}\right)^{2/3}}{10m}\frac{N_{tot}^{5/3}}{R^2}\text{ ;non-rel.} \\ \displaystyle\int_0^{p_f} pc\frac{V}{\pi^2\hbar^3}p^2\,dp = \frac{Vc}{\pi^2\hbar^3}\frac{p_f^4}{4} = \frac{3}{4}N_{tot}cp_f = \frac{3}{4}\hbar c N_{tot}\left(3\pi^2\right)^{1/3}\left(\frac{N_{tot}}{V}\right)^{1/3} = \frac{3\hbar c\left(\frac{9\pi}{4}\right)^{1/3}}{4}\frac{N_{tot}^{4/3}}{R}\text{ ; ultra-relativistic} \end{cases}$$

White Dwarf Stability

- Pressure:

$$P = -\frac{dE_{tot}}{dV} = \begin{cases} -\frac{3\hbar^2}{10m} N_{tot}^{5/3} \left(3\pi^2\right)^{2/3} \frac{d}{dV} V^{-2/3} = \frac{2\hbar^2}{10m} N_{tot}^{5/3} \left(3\pi^2\right)^{2/3} V^{-5/3} = \frac{\left(3\pi^2\right)^{2/3} \hbar^2}{5m} n^{5/3} & \text{;non - rel.} \\ -\frac{3\hbar c}{4} N_{tot}^{4/3} \left(3\pi^2\right)^{1/3} \frac{d}{dV} V^{-1/3} = \frac{\hbar c}{4} N_{tot}^{4/3} \left(3\pi^2\right)^{1/3} V^{-4/3} = \frac{\left(3\pi^2\right)^{1/3} \hbar c}{4} n^{4/3} & \text{; ultra - rel.} \end{cases}$$

- Compare:

$$P(0) \approx \frac{2\pi G \rho^2 R^2}{3} = \left\{ \begin{matrix} \frac{\left(3\pi^2\right)^{2/3} \hbar^2}{5m_e} \left(\frac{\rho}{2m_N}\right)^{5/3} & \text{;non - rel.} \\ \frac{\left(3\pi^2\right)^{1/3} \hbar c}{4} \left(\frac{\rho}{2m_N}\right)^{4/3} & \text{;ultra - rel.} \end{matrix} \right\} \Rightarrow R^2 = \begin{cases} \frac{3\left(3\pi^2\right)^{2/3} \hbar^2}{5m_e 2\pi G \left(2m_N\right)^{5/3}} \rho^{-1/3} \propto \frac{R}{M^{1/3}} \Rightarrow \quad R^3 \propto \frac{1}{M} \\ \frac{3\left(3\pi^2\right)^{1/3} \hbar c}{8\pi G \left(2m_N\right)^{4/3}} \rho^{-2/3} \propto \frac{R^2}{M^{2/3}} \Rightarrow \quad M_{\max} \quad ! \end{cases}$$

White Dwarf Stability

- If *R* **de**creases, gravitational energy more negative:

$$\frac{dV_{pot}^{grav}}{d(-R)} = -\frac{d}{dR}\left(-\frac{3GM^2}{5R}\right) = -\frac{3GM^2}{5R^2}$$

- …while kinetic energy goes up:

$$\frac{dE_{tot}^{kin}}{d(-R)} = -\frac{d}{dR}\left(\frac{3\hbar^2}{10m}\left(\frac{9\pi}{4}\right)^{2/3}\frac{N_{tot}^{5/3}}{R^2}\right) = \frac{3\hbar^2}{5m}\left(\frac{9\pi}{4}\right)^{2/3}\frac{N_{tot}^{5/3}}{R^3}\text{; non-rel.}$$

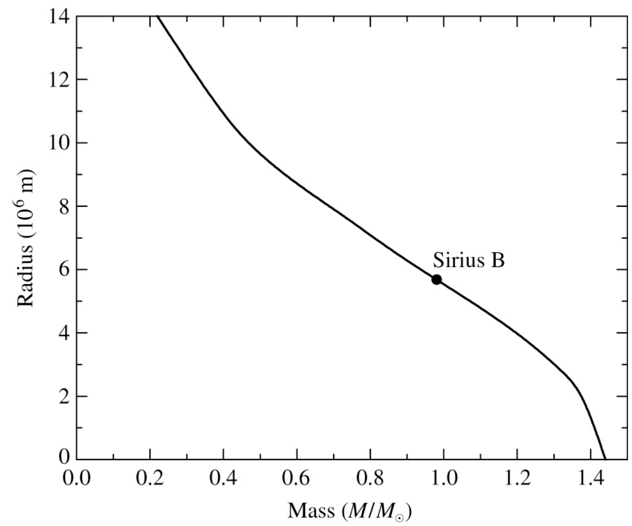
- Compare: Equilibrium if sum of derivatives = 0

$$-\frac{3GM^2}{5R^2} + \frac{3\hbar^2}{5m}\left(\frac{9\pi}{4}\right)^{2/3}\frac{N_{tot}^{5/3}}{R^3} = 0 \Rightarrow \quad R = \frac{\hbar^2 N_{tot}^{5/3}}{m_e GM^2}\left(\frac{9\pi}{4}\right)^{2/3} \propto \frac{M^{5/3}}{M^{6/3}}$$

=> Chandrasekhar Limit

- For less massive, larger white dwarfs:
  - $R \approx 5600$ km $(M/M_{sun})^{-1/3}$ => $V \propto 1/M$; $\rho \propto M^2$
  - $p_f = 670$ keV/c x $(n/n_{SiriusB})^{1/3} = 670$ keV/c x $(M/M_{sun})^{2/3}$

- as mass increases, gas becomes more and more relativistic and radius becomes even smaller => runaway collapse ($R \propto M^{-\infty}$)



- Mass limit $M_{ch} = 1.4\ M_{sun}$

- Above that mass (for a stellar remnant after blowing off outer hull) electron Fermi gas pressure not sufficient for stability -> neutron Fermi gas (see later)