Mathematical Modeling in Unusual Contexts: Rainbows, Halos and Glories.

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The subject of meteorological optics is a fascinating one; it includes the study of the above-mentioned phenomena as well as others too numerous to list here (a superb resource for all these topics, containing many wonderful photographs, is Dr. Les Cowley's website at http://www.sundog.clara.co.uk/atoptics/phenom.htm). Obviously there is some physics involved in the explanation of these phenomena, but fortunately it is not necessary to go into a lot of physical detail in order to appreciate the use of trigonometry and elementary calculus used in modeling these beautiful arcs on the sky. This article is presented as a possible resource for high school mathematics teachers who are looking for non-traditional contexts from which to teach some of the traditional precalculus and calculus topics. It has been field-tested several times, within the context of a college class that contained more advanced material, but the student excitement about and involvement with the subject matter was quite high once students had actually noticed some rainbows, halos and glories (and also sundogs) for themselves! [*The figures have yet to be added.*]

Rainbows

A thorough account of what might be called the "ray theory" for the formation of rainbows was provided in an NCTM publication, so it will not be necessary to reproduce the details of the article here. The primary rainbow is caused by light from the sun entering the observer's eye after it has undergone one reflection and two refractions in myriads of raindrops (see Figure 1). Using elementary geometry it is seen that the ray has undergone a total deviation of $D(i)$ radians, where $D(i) = \pi + 2i - 4r(i)$, in terms of the angles of incidence (i) and reflection (r) respectively. The latter is a function of the former, this relationship being expressed in terms of Snell's famous law of refraction $\sin i = n \sin r$, n being the relative index of refraction (of water, in this case). This relative index is defined as

$$n = \frac{\text{speed of light in medium I (air)}}{\text{speed of light in medium II (water)}} > 1.$$

Since the speed of light in air is almost that 'in vacuo', we will refer to n for simplicity as *the* refractive index; its generic value for water is $n \approx 4/3$, but it does depend slightly on wavelength (this is the phenomenon of dispersion, and without it we would only have bright "whitebows".) By examining the behavior of $D(i)$ it is readily established that for $i \in [0, \pi/2]$ the condition for an extremum, $D'(i) = 0$ implies that the critical angle of incidence i_c corresponding to an extremum of $D(i)$ is given by

$$i_c = \arccos\left(\left[\frac{n^2-1}{3}\right]^{1/2}\right). \tag{1}$$

Although realistically $k \leq 2$ (for reasons mentioned below), for k internal reflections the corresponding result is

$$i_c = \arccos\left(\left[\frac{n^2-1}{k(k+2)}\right]^{1/2}\right), \tag{2}$$

as is readily verified. Furthermore, for general k it can be shown that

$$D_k''(i_c) = \frac{2(k+1)(n^2-1)\tan r}{n^2\cos^2 r} > 0, \tag{3}$$

which is positive since $n > 1$ and $0 < r < \frac{\pi}{2}$, ignoring the special case of normal incidence, meaning that the clustering of deviated rays corresponds to a *minimum* deflection $D(i_c)$ (for $k = 1$ in particular). Note, however that the formulation for $k = 2$ in the above reference used, in effect, $-D$ to find a corresponding maximum deflection for this quantity. When $k = 1,\ D(i)$ may be written with sole dependence on i as

$$D(i) = \pi + 2i - 4\arcsin\left(\frac{\sin i}{n}\right). \tag{4}$$

Continuing with the case $k = 1$, some numerical values are in order. For a "generic" monochromatic rainbow (the whitebow referred to above), the choice $n = 4/3$ yields

$$i_c \approx \arccos\sqrt{\frac{7}{27}} \approx 59.4° \text{ and } D(i_c) \approx 138°.$$

The supplement of this angle, $180° - D(i_c) \approx 42°$ is the semi-angle of the rainbow 'cone' formed with apex at the observer's eye, the axis being along the line joining the eye to the antisolar point.

Next we mention dispersion, and we can also study this in an elementary way. While not exactly standardized, the visible part of the electromagnetic spectrum extends from the 'red' end (about 700 – 650 nm) to the 'violet' end (about 425 – 400 nm), one nanometer (nm) being 10^{-9} m. For red light of wavelength $\lambda = 656.3$ nm, the refractive index $n \approx 1.3318,$ whereas for violet light of wavelength $\lambda = 404.7$ nm, the refractive index $n \approx 1.3435;$ a slight but very significant difference! All that has to be done is the calculation of i_c and $D(i_c)$ for these two extremes of the visible spectrum, and the difference computed, and voilà! We have found the angular width of the primary rainbow. In fact since $D(i_c) \approx 137.75°$ and $139.42°$ for the red and violet ends, respectively, the angular width $\Delta D \approx 1.67° \approx 1.7°$, or about three full moon angular widths.

Now we consider the secondary rainbow In this case there is an additional reflection, so the geometry yields a total ray deviation of

$$D(i) = 2\pi + 2i - 6r = 2i - 6r \text{ (modulo } 2\pi) = 2i - 6\arcsin\left(\frac{\sin i}{n}\right). \tag{5}$$

Proceeding as before we find that

$$i_c = \arccos\left(\left[\frac{n^2-1}{8}\right]^{1/2}\right), \tag{6}$$

as is evident from equation (2). For our generic n-value of 4/3,

$$\cos i_c = \sqrt{\frac{7}{72}} \approx 0.3118, \text{ so } i_c \approx 71.8°. \text{ Then}$$

$$D(i_c) = 2i_c - 6\arcsin(0.7125) = 143.6° - 276.6° = -129°.$$

The negative sign is of no significance here, since the ray geometry is cylindrically symmetric about the central axis, but for this reason sometimes $D(i)$ is defined as the negative of the expression used here, so be careful which one you are using if you calculate $D''(i_c)$. Just as the supplement of $D(i_c)$ for the primary bow is approximately $42°,$ so for secondary bow the supplementary angle is 51°; thus the secondary bow is about 9° higher in the sky than the primary bow.

In general, for k internal reflections it can be shown that

$$D(i) = k\pi + 2i - 2(k+1)\arcsin\left(\frac{\sin i}{n}\right), \tag{7}$$

and after an interesting trigonometric calculation, in particular it can be shown that when $k = 1$,

$$D(i_c) = 2\arccos\left\{\frac{1}{n^2}\left(\frac{4-n^2}{3}\right)^{3/2}\right\}, \tag{8}$$

thus expressing the angle of minimum deviation (the "rainbow angle") in terms of the refractive index alone. In principle this can also be done for higher-order rainbows, although the algebra gets messy. Substituting $n = 4/3$ into the above equation gives

$$D(i_c) = 2\arccos\left\{\frac{9}{16}\left(\frac{20}{27}\right)^{3/2}\right\} = 2\arccos\{0.3586\} \approx 138°,$$

thus recovering the result for the primary bow. Regarding the tertiary and higher bows: these bows are rarely, if ever seen, so don't look for them; the tertiary is too close to the sun and the higher orders are probably too faint!

Ice Crystal Halos

Halos are formed when sunlight is refracted, reflected or both from ice crystals in the upper atmosphere and enters the eye of the observer, but even the common types are easily missed. They are usually produced when a thin uniform layer of cirrus or cirrostratus cloud covers large portions of the sky, especially in the vicinity of the sun. Surprisingly, perhaps, they may occur at any time of the year, even during high summer, because above an altitude of about 10 km it is always cold enough for ice crystals to form. Very many of these crystals are hexagonal prisms; some are thin flat plates while others are long columns, and sometimes the latter have bullet-like or pencil-like ends. A significant feature of all these crystals is that while any given type may have a range of sizes, the angles between the faces are the same. Although they do not possess *perfect* hexagonal symmetry, of course, they are sufficiently close to this that simple geometry based on such idealized forms suffices to describe the many different arcs and halos that are associated with them. The halo formation that results from cirrus cloud crystals depends on two major factors: their shape and their orientation as they fall. Their shape is determined to a great extent by their history, i.e. the temperature of the regions through which they drift as they are drawn down by gravity and buffeted around by winds and convection currents.

Generally, halos may be seen several times a month in Europe and parts of the United States. The most frequent is the 22° circular halo, followed by parhelia (sundogs) and then the upper tangent arc. In my own rather limited experience, the sundogs are most common (at least as I walk to work!). In order to explain the reasons for the various angles, e.g. 22°, etc., it is necessary to examine the crystal geometry in more detail. There are several other fairly common halos and arcs to be seen, depending on the orientation of the ray relative to the crystal, but we focus here solely on the 22° halo. First, however, it is necessary to state things a little more formally in terms of theorems (presented here without proof) concerning the refraction of light through prisms of apex angle γ and relative (to air) refractive index n.

Theorem 1: *The deviation or deflection angle for light refracted through a prism is a minimum for symmetric ray paths.*

Theorem 2: *The minimum deviation angle D_m for a prism satisfies the relation*

$$n = \frac{\sin[\frac{1}{2}(\gamma + D_m)]}{\sin\frac{1}{2}\gamma}. \tag{9}$$

From this result we may solve for D_m to obtain

$$D_m = 2\arcsin(n\sin\frac{\gamma}{2}) - \gamma. \tag{10}$$

We can use this result to explain the occurrence of the 22° halo. There are in fact three prism angles in a hexagonal ice crystal prism: 60° (light entering side 1 and exiting side 3); 90° (light entering a top or bottom face and exiting through a side) and 120° (light entering side 1 and being totally internally reflected by side 2), but our concern here is with the first one only. The refractive index of ice for yellow light is $n \approx 1.31.$ For the apex angle $\gamma = 60°,$ the minimum deviation is

$$D_m = 2\arcsin(1.31\sin 30°) - 60° = 21.8° \approx 22°,$$

For angle of incidence i, the deviation $D(i)$ may also be written in terms of the constants of the problem for a particular crystal, n and $\gamma,$ thus

$$D(i;\gamma,n) = i - \gamma + \arcsin\left(n\sin\left[\gamma - \arcsin\left(\frac{\sin i}{n}\right)\right]\right). \tag{11}$$

All of this goes to show, of course, that not only are ice crystals responsible for some magnificent displays in the sky from time to time, but also they are associated with some rather impressive looking transcendental equations! It is the case for the ray path sketched in figure 2 that the minimum value D_m in occurs near 22° of arc. As in the case of the rainbow, all possible deviations are present in reality, but it is the “clustering” of deviated rays near the minimum that provides the observed intensity in the halos (but unlike the case of the rainbow, no reflection contributes to their formation in these two cases). One way to verify analytically that there is a true minimum is of course to show from equation (11) that when $D'(i) = 0,\ D''(i) > 0$, but this is left as an interesting exercise; it’s not as bad as it looks. There are restrictions on the angle of incidence i such that outside these, no value of D can be defined for real parameters n and $\gamma.$ These restrictions arise because of the requirement from equation (11) that, in particular,

$$\left(n\sin\left[\gamma - \arcsin\left(\frac{\sin i}{n}\right)\right]\right) \leq 1,$$

(i lying within the first quadrant). This inequality places a lower bound on α by unfolding expression (11) to obtain

$$i \geq \arcsin\left(n\sin\left[\gamma - \arcsin\left(\frac{1}{n}\right)\right]\right),$$

where it should be recalled that the ‘sin’ function is a one-to-one function in the range $(0,\pi/2),$ so that the ‘arcsin’ function is monotone increasing in its domain. For $\gamma = 60°,$ this corresponds to $i \geq 13.5°$. Other arguments enable limits to be placed on the sun’s altitude or the latitude of the observer for certain types of halo to be visible (atmospheric conditions permitting). We illustrate this for the *circumzenithal arc*.

The circumzenithal arc

The circumzenithal arc is a circular halo that is centered on the zenith point of the observer.

According to Les Cowley's website: *"The circumzenithal arc is the most beautiful of all the halos. The first sighting is always a surprise, an ethereal rainbow fled from its watery origins and wrapped improbably about the zenith. It is often described as an "upside down rainbow" by first timers. Someone also charmingly likened it to "a grin in the sky". Look straight up near to the zenith when the sun if fairly low and especially if sundogs are visible. The centre of the bow always sunwards and red is on the outside."*

Now for a little geometry and trigonometry. The sunlight enters from the top of the ice crystal (as shown in Figure 3) and exits from the side. We can derive a condition on the sun's altitude θ for the halo to be visible, i.e. θ must be greater than (or less than) a certain angle, and will assume, as before, that the refractive index of the ice is 1.31. From Figure 3 we can see that no ray will enter the observer's eye if total internal reflection takes place at B. This places restrictions on the angle of internal incidence (measured from the normal to the vertical face) θ_2 and hence on the external angle of incidence to the horizontal face, θ_1. Total internal reflection will occur if $r_2 \geq \pi/2$, so taking the equality as the limiting case we have

$$\sin\theta_2 = n_2 \sin r_2 = n_2 = n_1^{-1} = (1.31)^{-1} \approx 0.76,$$

since this is an ice-to-air interface. Then

$$\theta_2 \geq \arcsin(0.76) \approx 49.7°.$$

Therefore $r_1 \leq 40.3°$, and for the air-ice interface, $\sin\theta_1 = n_1 \sin r_1$, so total internal reflection occurs when

$$\theta_1 \leq \arcsin[1.31 \sin(40.3°)] \approx 57.9°.$$

Under these circumstances no ray enters the observer's eye, and therefore the circumzenithal arc can only be seen when $\theta_1 \geq 57.9°$, i.e. when the sun's altitude θ $(= 90° - \theta_1)$ is less than about 32°. Similar kinds of calculation apply to other halo types to yield the corresponding limitations on solar elevation (and hence latitudes of visibility) or perhaps crystal orientation.

The glory

Mountaineers and hill-climbers have noticed on occasion that when they stand with their backs to the low-lying sun and look into a thick mist below them, they may sometimes see a set of colored circular rings (or arcs thereof) surrounding the shadow of their heads. Although an individual may see the shadow of a companion, the observer will see the rings only around his or her head. Again, many details of glories can be found from Les Cowley's website. While not noted as frequently as the rainbow, this phenomenon may be seen most commonly from the air, with the glory surrounding the shadow of the airplane. Once an observer has seen the glory, if looked for, it is readily found on many subsequent flights (provided one is on the shadow side of the aircraft!).

The phenomenon can be understood in the simplest terms as essentially the result of light backscattered by cloud droplets, the light undergoing some unusual transformations en route to the observer (with a correspondingly complicated mathematical description), transformations that are *not* predictable by standard geometrical optics, unlike the basic description of the rainbow. That this must be the case is easily demonstrated by noting a fallacy present in at least one popular meteorological text. The glory, it is claimed, is formed as a result of a ray of light tangentially incident on a spherical raindrop being refracted into the drop, reflected from the back surface and reemerging from the drop in an exactly antiparallel direction into the eye of the observer (see Figure

4(a)). If such a picture is correct, then since the angle of incidence of the ray is 90°, it follows by Snell's law of refraction that the angle of refraction is

$$r = \arcsin\left(\frac{1}{n}\right)$$

where n is the refractive index of the raindrop. For an air/water boundary, $n \approx 4/3$ (ignoring the effects of dispersion here, though this does occur as noted above; note also that for a water/air boundary the reciprocal of n must be used), and so $r \approx 48.6°$. This means (by the law of reflection) that at the back of the drop, the ray is deviated by more than a right angle, since $2r \approx 97.2°$, and by symmetry the angle of incidence *within* the drop for the exiting ray is also $48.6°$, so the total deviation angle (as we saw for the primary rainbow) is

$$D(i) = \pi + 2i - 4\arcsin\left(\frac{\sin i}{n}\right) = -4\arcsin\left(\frac{3}{4}\right) \approx -194.4° \text{ or } +165.6°$$

(modulo 2π) since $i = 90°$. This means that the exiting ray is about 14° short of being "antiparallel". It just won't work as a mechanism for the glory! There are basically two potential ways out of this. We could ask what value of refractive index n would be necessary for the diagram to be correct; thus $i = 90°$ as before but now $r = 45°$; this means that

$$n = \frac{\sin i}{\sin r} = \sqrt{2} \approx 1.4,$$

i.e. between that of water and glass! What other option remains? One possibility is that somehow the ray travels around the surface (as a surface wave) for part (or parts) of its "trip", the surface portion comprising the missing piece θ, where from Figure 4(b) (drawn for the symmetric case only)

$$\theta = 180° - 2(180° - 2r) = 4r - 180° \approx 14.4°$$

for $r \approx 48.6°$. The resulting path in the droplet need not be symmetric to account for an antiparallel exiting ray.

This article presents some of the basic mathematical ideas behind some common meteorological phenomena, though sundogs have not been addressed here, the theory behind them is very similar to that for the 22° halo, except that they are formed by light being refracted through horizontally oriented hexagonal ice crystals. The analysis presented here is not, of course, new: it can be culled from many sources because the subject of meteorological optics has been around for a long time! What *is* new, it is hoped, is the presentation of these ideas as a potential topic for mathematical modeling in the high school classroom. It should also be noted that many subtle features associated with such optical effects in the atmosphere require much more powerful mathematical tools to explain them. Thus a detailed study of the glory or of *supernumerary rainbows* is, regrettably, far too complicated to go into here, but the interested reader is referred to the bibliography for references and further details (and also to Philip Laven's excellent website *http://www.philiplaven.com/index1.html*).

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Figure Captions

Figure 1: the path of a ray of sunlight inside a spherical raindrop which contributes to the formation of the primary rainbow upon exiting the drop.

Figure 2: a hexagonal ice crystal showing one possible (symmetric) ray path through the crystal and the subsequent 22° minimum deviation corresponding to the 22° halo.

Figure 3: a ray path for the circumzenithal arc; it enters through a horizontal face of a hexagonal prism, and exits through a vertical side. The dotted ray path inside the crystal indicates total internal reflection.

Figure 4: (a) An incorrect (but commonly drawn) ray path in a spherical cloud droplet alleged to contribute to the glory. Although the glory is essentially a backscattering phenomenon, it cannot be produced exactly as shown, because a tangential incident ray will not be returned antiparallel to its original path, as demonstrated in the text. Figure (b) illustrates one possible correct path; it involves the ray traveling as a surface wave around a portion of the droplet surface for a total of about 14° (a symmetric path is shown).